Random Walks on Graphs : Assignment 2

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Submit solutions to Q2, Q5, Q8, Q9 on Moodle by 2nd March 10:00 PM.

The following problems concern Markov chains on a finite state space V.

1. Let π_0 be an arbitrary probability distribution on V. Define
$$\nu_n = \frac{1}{n}(\pi_0 + \pi_0 P + \ldots + \pi_0 P^{n-1}).$$
Show the following :
   (a) Show that for any $x \in V$, $|\nu_n P(x) - \nu_n(x)| \leq 2n^{-1}$.
   (b) Show that there exists a subsequence n_k such that $\nu_{n_k}(x)$ converges for every $x \in V$.
   (c) Show that the subsequential limit above is a stationary distribution.

2. Give a direct proof that stationary distribution for an irreducible Markov chain is unique and strictly positive if it exists.
Hint : *If π_1, π_2 are two stationary distributions, show that $\frac{\pi_1(x)}{\pi_2(x)} = \frac{\pi_1(y)}{\pi_2(y)}$ if $P(x,y) > 0$ and $\frac{\pi_1(x)}{\pi_2(x)}$ minimizes the ratio. A similar argument using $P(x,y) > 0$ also applies for strict positivity.*

3. For a subset $A \subset V$, define $f(x) = \mathbb{E}_x(\tau_A)$. Show that f is uniquely determined by the following two equations
$$f(x) = 0, x \in A; \quad f(x) = 1 + \sum_{y \in V} f(y) P(x,y), x \notin A.$$

4. For $x \neq a$ justify in detail that $\mathbb{P}_a(\tau_z < \tau_a^+ | X_1 = x) = \mathbb{P}_x(\tau_z < \tau_a^+)$.

5. Show that effective resistances form a metric on any graph.

6. Verify the reductions under series and parallel laws are valid i.e., the effective resistance is unchanged.

7. Suppose A, Z are disjoint subsets of V. Define $C(A \leftrightarrow Z) := C(a \leftrightarrow Z)$ by identifying A to a single vertex a [1]. Show that if h is the voltage applied at A, Z such that $h_{|A}, h_{|Z}$ are constants then
$$h_{|A} - h_{|Z} = I(A,Z) R(A \leftrightarrow Z),$$
where $I(A,Z) := \sum_{x \in A} \sum_{y \in Z} I(x,y)$.

[1] $C(a \leftrightarrow Z)$ will be defined in class soon

For the following problems, let (G, μ) be a locally finite, connected, infinite vertex, weighted graph.

8. For $z \in V$, let $\tau_z = \min\{n \geq 0 : X_n = z\}$ and $\tau_z^+ = \min\{n \geq 1 : X_n = z\}$. Show that the following conditions are equivalent:
   - (T1) $\exists x \in V$ such that $\mathbb{P}^x(\tau_x^+ < \infty) < 1$.
   - (T2) For all $x \in V$, $\mathbb{P}^x(\tau_x^+ < \infty) < 1$.
   - (T3) For all $x \in V$, $\sum_{n=0}^{\infty} \mathbb{P}^x(X_n = x) < \infty$.
   - (T4) For all $x, y \in V$ with $x \neq y$, either $\mathbb{P}^x(\tau_y < \infty) < 1$ or $\mathbb{P}^y(\tau_x < \infty) < 1$.
   - (T5) For all $x, y \in V$ with $x \neq y$, $\mathbb{P}^x(\sum_{n=0}^{\infty} 1(X_n = y) < \infty) = 1$

9. Show that the random walk on (G, μ) has a stationary probability distribution iff $\sum_{x,y \in V} \mu(x, y) < \infty$.

10. Let G_n be a sequence of finite induced sub-graphs of G that *exhaust* G i.e., $G_n \subset G_{n+1}$ and $G = \cup_n G_n$. Let $Z_n = V(G) \setminus V(G_n)$ and G_n^W is the graph formed from G by identifying Z_n with a single vertex z_n. Show that $\lim_{n\to\infty} C(a \leftrightarrow z_n)$ is the same for any sequence of induced subgraphs G_n that exhaust G. Here $C(a \leftrightarrow z_n)$ is the effective resistance between a to z_n in the graph G_n^W.