Random Walks on Graphs : Assignment 3

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Submit solutions to Q1, Q3, Q6, Q8 on Moodle by 16th March 10:00 PM.

The following problems concern Markov chains on a finite state space V with P as the stochastic matrix and π as its stationary distribution.

1. Let $\mathcal{P}(V)$ be the space of probability distributions on V. Show that
$$d(t) := \sup_{\mu \in \mathcal{P}(V)} \|\mu P^t - \pi\|_{TV},$$
$$\bar{d}(t) := \sup_{\mu, \nu \in \mathcal{P}(V)} \|\mu P^t - \nu P^t\|_{TV}.$$

2. Show that for $t, s \geq 0$, $\bar{d}(t+s) \leq \bar{d}(t)\bar{d}(t)$ and $d(t+s) \leq d(t)\bar{d}(s)$.

3. Show that for $f, g : V \to \mathbb{R}$ and $t \geq 0$,
$$\mathbb{COV}_\pi(P^t f, g) \leq (1-\gamma_*)^t \sqrt{\mathbb{VAR}_\pi(f)\mathbb{VAR}_\pi(g)},$$
where γ_* is the absolute spectral gap.

4. Suppose that P is reversible. Show that $P^{2t+2}(x,x) \leq P^{2t}(x,x)$.

5. Use Q.3 to prove the following result known as the expander mixing lemma. Let $G = (V, E)$ be a d-regular graph with n vertices and let β be the largest eigenvalue of the adjacency matrix of G. Let $e(A, B) = |(A \times B) \cap E|$. Show that
$$|e(A,B) - \frac{d|A||B|}{n}| \leq \beta\sqrt{|A||B|}.$$

6. Let X_t, Y_t be two Markov chains with transition matrix P but with different initial distributions. A **Markovian coupling** of X_t, Y_t is a Markov chain (X'_t, Y'_t) on $V \times V$ such that it satisfies for all x, y, x', y'
$$\mathbb{P}(X'_{t+1} = x' | X'_t = x, Y'_t = y) = P(x, x'),$$
$$\mathbb{P}(Y'_{t+1} = y' | X'_t = x, Y'_t = y) = P(y, y').$$
Show that there exists a Markovian coupling (X'_t, Y'_t) such that
$$\text{if } X'_t = Y'_t \text{ then } X'_s = Y'_s \text{ for all } s \geq t. \tag{1}$$
Assume now that P is an irreducible and aperiodic transition matrix. Show that if X'_t, Y'_t are independent Markov chains with same distribution as the Markov chains X_t, Y_t respectively then $\mathbb{P}(X'_t = Y'_t \text{ for some } t) = 1$.

7. Given a Markovian coupling (X'_t, Y'_t) satisfying (1), define $\tau_c := \inf\{t : X'_t = Y'_t\}$. Suppose that $X'_0 \overset{d}{=} \mu, Y'_0 \overset{d}{=} \nu$. Then show that

$$\|\mu P^t - \nu P^t\|_{TV} \leq \mathbb{P}(\tau_c > t).$$

Use this to show that $\|\mu P^t - \pi\|_{TV} \to 0$ as $t \to \infty$.

8. Let P be the transition matrix corresponding to the lazy random walk on the d-dimensional Torus $\mathbb{Z}_n^d := (\mathbb{Z}/n\mathbb{Z})^d$.

   We define the following Markovian coupling X_t, Y_t of lazy random walks starting at $x, y \in \mathbb{Z}_n^d$ respectively. First one of the d-coordinates is chosen at random. It is the same coordinate for both the walks. Say that i is the common co-ordinate. We now update as follows : If $X_t(i) = Y_t(i)$, then both of them move by $+1, -1$ or 0 in the ith co-ordinate with probabilities $1/4, 1/4, 1/2$ respectively. If $X_t(i) \neq Y_t(i)$, we randomly choose one of the two chains to move and leave the other fixed. Suppose we decide that $Y_t(i)$ is to be moved. Then $Y_t(i)$ is moved by $+1$ or -1 with probabilities $1/2, 1/2$ respectively. Define $\tau_i := \inf\{t : X_t(i) = Y_t(i)\}$.

   Show the following :

   (a) $\mathbb{P}(\tau_1 > t) \leq 4^{-k}$ for $t \geq kdn^2$.

   (b) $t_{mix}(\epsilon) \leq dn^2 \lceil \log_4(d/\epsilon) \rceil$ for $\epsilon < 1/2$.

9. Compute the relaxation time for the simple random walk and lazy random walk on the following graphs : Path graph P_n, Complete graph K_n and complete bi-partite graph $K_{m,n}$.

10. If the bounds for mixing time derived via relaxation time is not tight in the above Markov chains, can you improve the upper bound via coupling ?

    NOTE : This is an open ended question more for some of you to think of coupling method.