Random Walks on Graphs : Assignment 4

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Submit solutions to Q1, Q2 and Q8 on Moodle by 6th April 10:00 PM.
The following problems concern Markov chains on a finite state space V.

1. Let P and $\tilde{P}$ be two reversible and irreducible stochastic matrices with stationary distributions π and $\tilde{\pi}$ respectively. Let Γ_{xy} be the set of all paths from x to y in $E = \{(x,y) : P(x,y) > 0\}$ and ν_{xy} be a probability measure on Γ_{xy}. Define the congestion ratio as

$$B := \max_{e \in E} \left(\frac{1}{Q(e)} \sum_{(x,y) \in \tilde{E}} \tilde{Q}(x,y) \sum_{\gamma : e \in \gamma \in \Gamma_{xy}} \nu_{xy}(\gamma)|\gamma| \right).$$

   Show that $\tilde{\gamma}_2 \leq c(\pi, \tilde{\pi}) B \gamma_2$.

2. Let p and $\tilde{p}$ be two increment probability measures of a reversible and irreducible stochastic matrices on a finite group G. Let S and $\tilde{S}$ be the respective symmetric generating sets. Let $P_a := \{(s_1, \ldots, s_k) : a = s_1 \ldots s_k, s_i \in S\}$ be all the possible ways to write a in terms of elements of S and ν_a, a probability measure on P_a and $N(s,\gamma)$ be the number of times s occurs in $\gamma \in P_a$. Show that $\tilde{\gamma}_2 \leq B\gamma_2$ where

$$B := \max_{s \in S} \left(\frac{1}{p(s)} \sum_{a \in \tilde{S}} \tilde{p}(a) \sum_{\gamma \in P_a} \nu_a(\gamma) N(s,\gamma)|\gamma| \right).$$

For the following problems (G, μ) be a locally finite, connected weighted graph on a infinite vertex set V.

3. Show that $C_o(V) = \{f : V \to \mathbb{R} : \mathrm{Supp}(f) \text{ is finite }\}$ is dense in $L^p(V,\mu)$ for all $p \in [1, \infty]$.

4. For all $p, r \in [1, \infty]$, let $A : L^p(V,\mu) \to L^r(V,\mu)$. Show that

$$\| A \|_{p \to r} := \sup\{\|Af\|_r : \|f\|_p \leq 1\}$$

   is a norm.

5. Let $C(V) = \{f : V \to \mathbb{R}\}$. Show that $P_n : C(V) \to C(V)$ defined by

$$P_n f(x) = \sum_{n=0}^{\infty} \mathbb{P}_x(X_n = y) f(y),$$

   satisfies $P_n = (P_1)^n$.

6. Let $H^2(V) = \{f \in C(V) : \mathcal{E}(f,f) < \infty\}$ where

$$\mathcal{E}(f,g) = \frac{1}{2} \sum_{x \in V} \sum_{y \in V} \mu(x,y)(f(x) - f(y))(g(x) - g(y)) \text{ and } \|f\|_{H^2} = \mathcal{E}(f,f) + f(o)^2,$$

   for $f, g \in C(V)$ and (fixed) $o \in V$.

   (a) Show that convergence in H^2 implies pointwise convergence.

   (b) Show that $H^2(V)$ is a Hilbert space. Please specify the inner product.

   (c) Let $f_n \in H^2$ with $\sup_n \| f_n \|_{H^2} < \infty$. Then there exists $\{f_{n_k}\}$ and $f \in H^2$ such that for each $x \in V$,

$$\lim_{k \to \infty} f_{n_k}(x) = f(x) \text{ and } |f\|_{H^2} \leq \liminf_{k \to \infty} \| f_{n_k} \|_{H^2}$$

(d) The above conclusion also holds if A_n is an increasing sequence of finite sets such that $\cup A_n = V, O \in A_1$ and $f_n : A_n \to \mathbb{R}$ with $\sup_n(|f_n(O)|^2 + \mathcal{E}_{A_n}(f)) < \infty$ where for any finite $A \subset V$, we define

$$\mathcal{E}_A(f,g) := \mathcal{E}(f1_A, g1_A).$$

7. Let $\mathcal{L}$ be the Laplacian operator. Show the following.

   (a) $\mathcal{L} : H^2 \to \mathbb{L}^2$ is well-defined and $\|\mathcal{L}f\|^2 \leq 2\mathcal{E}(f) \leq 2\|f\|_{H^2}$.

   (b) For $f \in H^2$ and $g \in \mathbb{L}^2$, show that $\mathcal{E}(f,g) = \langle -\mathcal{L}f, g\rangle$.

   (c) Show that $\mathcal{L}$ is a self-adjoint operator on $\mathbb{L}^2$ and

$$\langle -\mathcal{L}f, g\rangle = \langle f, -\mathcal{L}g\rangle.$$

8. Let $H^2_0(V)$ be the closure of $C_0(V)$ in H^2.

   (a) Show that $\mathbb{L}^2 \subset H^2_0 \subset H^2$ and all three containements can be strict.

   (b) Show that if $f \in H^2_0$ then $f_+, f_- \in H^2_0$.

   (c) Show that if $f \in H^2_0, f \geq 0$ and $f_n \to f$ in H^2 then $\min(f_n, f) \to f$ in H^2.

   (d) Show that $H^2_0(\mathbb{Z}_+) = H^2(\mathbb{Z}_+)$ where $\mathbb{Z}_+$ is the non-negative integer considered as the induced subgraph of the usual integer subgraph.