Random Walks on Graphs : Assignment 5

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Submit solutions to Q4, Q7 and Q8 on Moodle by 15th April 10:00 PM.

For the following problems (G, μ) be a locally finite, connected weighted graph on a infinite vertex set V.

1. Show the following properties of effective resistance.
   (a) If $A \subset A', B \subset B'$ with $A' \cap B' = \emptyset$ then $R_{eff}(A', B') \leq R_{eff}(A, B)$.
   (b) Show that *shorting* (i.e., identifying $x = y$ for an edge (x, y)) decreases effective resistance.
   (c) Show that *cutting* (i.e., removing an edge (x, y)) increases effective resistance.
2. If A^c is finite and $A \cap B = \emptyset$, show that
$$C_{eff}(B, A) = \sum_{x \in B} \mu(x) \mathbb{P}_x(\tau_A < \tau_B^+).$$
3. Show that if $f \in H^2, g := (0 \vee g) \wedge 1$ then $g \in H^2$ and $\mathcal{E}(g) \leq \mathcal{E}(f)$.
4. Let $V = \mathbb{Z}^3$ be equipped with the cannonical edges and natural weights. Let X_n be the random walk on it.
   (a) Show that X_n is transient on $\mathbb{Z}^3$.
   (b) Let $n \geq 1$, $A = \mathbb{Z}^3 \setminus \{0\}$. Let $h_n, h : V \to [0, 1]$ be given by
$$h_n(x) = \mathbb{P}_x(T_0 \geq n) = \mathbb{P}_x(X_k \in A, 1 \leq k \leq n)$$
and
$$h(x) = \mathbb{P}_x(T_0 = \infty) = \mathbb{P}_x(X_n \in A, \text{ for all } n \geq 0).$$
      i. Show that $h_n = Q^n 1_A$ and $h = Qh$, where Q is the restriction of P onto A.
      ii. Suppose $\alpha = \sup_{x \in A} h(x)$, show that $0 < \alpha \leq 1$ and $h \leq \alpha 1_A$
      iii. Using (i) and (ii), conclude that $h \leq \alpha h_n$
      iv. Conclude that $\max_{x \in \partial A} h(x) \neq \sup_{x \in \bar{A}} h(x)$.
5. Consider $\mathbb{Z}^d$ for $d \geq 1$ with natural weights and the random walk X_n on it. Let $A \subset \mathbb{Z}^d$.
   (a) Let $d = 1$, $a < 0 < b$, $A = (a, b)$ Show that $\mathbb{P}_0(\tau_{A^c} > n(b - a)) \leq (1 - \frac{1}{2^{b-a}})^n$.
   (b) Let $\emptyset \neq A \subset \mathbb{Z}^d$ such that $\mid A \mid < \infty$. Then show that for any $x \in \mathbb{Z}^3$,
$$\mathbb{P}_x(\tau_{A^c} > n) \leq c_1 \rho^n,$$
   for some $c_1 > 0$ and $0 < \rho < 1$.
6. (a) Show that $\mathbb{Z}^d, \mathbb{R}^d, [0, 1] \times \mathbb{R}^d$ are roughly isometric spaces.
   (b) Let G be a finitely generated infinite group, and λ, Λ' be two sets of generators. Let Γ, Γ' be the associated Cayley graphs. Then Γ, Γ' are roughly isometric.
7. Let $\alpha > 0$. Consider the graph $\mathbb{Z}_+$ with weights $\mu^\alpha(n, n + 1) = \alpha^n$.
   (a) Show the graph $(\mathbb{Z}_+, \mu^\alpha)$ has controlled weights. Does it have bounded weights ?
   (b) Show that the graph is recurrent if and only if $\alpha \leq 1$.
   (c) For $\alpha \neq \beta$ are $(\mathbb{Z}_+, \mu^\alpha)$ and $(\mathbb{Z}_+, \mu^\beta)$ roughly isometric ?

8. Consider the graph G obtained by identifying the origins of $\mathbb{Z}_+$ and $\mathbb{Z}^3$.

   (a) Show that G is transient.

   (b) Let $f_n : V \to \mathbb{R}$ be given by

   $$f_n(x) = \begin{cases} 1 - \frac{k}{n} & \text{if } x = k \text{ for } k \leq n \\ 0 & \text{otherwise} \end{cases}$$

   Find f such that $f_n \to f$ in H^2. Conclude that H^2, H_0^2 and L^2 are all distinct spaces.

   (c) Let $h : V \to \mathbb{R}$ be given by

   $$h(x) = \begin{cases} \mathbb{P}_x(\tau_0 < \infty) & \text{if } x \in \mathbb{Z}^3 \\ 1 + 6p_0 n & \text{if } x = n \in \mathbb{Z}_+, \end{cases}$$

   where $p_o = \mathbb{P}_0($ R.W on $\mathbb{Z}^3$ does not return to 0). Show that h is non-constant positive harmonic function on G.

   (d) Show that G satisfies the Liouville property.