Random Walks on Graphs : Assignment 7

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Submit solutions to Q1 Q2, Q3 and Q7 on Moodle by 13th May 10:00 PM.

For the following problems (G, μ) be a locally finite, connected weighted graph on a infinite vertex set V.

1. Show that the lamplighter graph on $\mathbb{Z}$, $LL(\mathbb{Z})$ has exponential growth but is amenable.

2. Show that join of $\mathbb{Z}^2$ with a binary tree $\mathbb{T}_2$ is amenable but ballistic.

3. Consider, B the join of two copies of $\mathbb{Z}^3$ at the origin with natural weights. Let $f : V \to \mathbb{R}$ be such that it is 1 on one copy and -1 on the other copy. Using f show that the weak Poincare inequatlity does not hold on G.

4. Let (G, μ) be a weighted graph. Let $f \in L^2(V)$. Then for any $n \geq 0$,

$$0 \leq \|P^{n+1} f\|_2^2 - \|P^{n+2} f\|_2^2 \leq \|P^n f\|_2^2 - \|P^{n+1} f\|_2^2$$

5. Let (G, μ) be a weighted graph. Let $f \in L^2(V)$. Then for any $n \geq 0$,

$$\|f\|_2^2 - \|P^n f\|_2^2 \leq 2n\mathcal{E}(f, f)$$

6. Suppose G is $\mathbb{Z}^d$ equipped with weights $\mu(x, y)$ such that $\mu(x, y) \geq \frac{c_1}{d}$ for some $c_1 > 0$. Let the heat kernel of the random walk on G be denoted by $p_n(x, y)$. Then show that

$$p_n(x, y) \leq c_2 n^{-\frac{d}{2}},$$

   for some $c_2 > 0$.

7. Let G be two copies of $\mathbb{Z}^d$ joined at the origin. Let the transition kernel of the random walk on G be denoted by $p_n(x, y)$. Then show that

$$p_n(x, y) \leq c_2 n^{-\frac{d}{2}},$$

   for some $c_2 > 0$.

8. Let G be a vertex transitive graph with natural weights with o denoting an arbitrary vertex. Assume that $|B(o, n)| \leq Ae^{Cn^\alpha}$ for $A, C \in (0, \infty)$ and $0 < \alpha \leq 1$. Show that $\mathbb{P}_o$ a.s. ,

$$\limsup \frac{d(o, X_n)}{n^{1/(2-\alpha)}} \leq (2C)^{1/(2-\alpha)}.$$

9. Let G be a vertex transitive graph with natural weights. Assume that $|B(o, n)| \leq Cn^2$ for some $C < \infty$. Show that the simple random walk on G is recurrent.