Quantum Field Theory: PH6418/ EP4618 (Fall 2022)

Notes for Lecture 1*

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1 Introduction

1. ***What is a field theory?***

   A field theory is a system of a infinite number of degrees of freedom, with the degrees of freedom labeled by one or more continuous parameters.

2. ***In which contexts/situations do field theories arise?***

   A. Fields can arise upon taking a well defined ***continuum limit*** **o**f a large number (discrete) of point particles or large number of degrees of freedom e.g. fluid dynamics from a dilute collection of particles or a solid from harmonic chains or collections of spins (spin chains).

   B. Fields can also arise in the absence of point particles (matter). For example, we need to introduce some kind of fields to describe the interaction between objects separated by some distance in space. We think of one of the objects creating a force field around itself and the second object feeling a force when it enters the field. E.g. the electric fields, $\boldsymbol{E}(\boldsymbol{x})$, or magnetic field, $\boldsymbol{B}(\boldsymbol{x})$, gravitational field, $\boldsymbol{g}(\boldsymbol{x})$ and the Higgs field, $\Phi(\boldsymbol{x})$. Here as well one can see the dynamical degree of freedom (electric or magnetic or gravitational or Higgs) is labeled by three continuous parameters, namely the position vector, $\boldsymbol{x}$.

2 Field theory arising from a continuum limit of the 1D harmonic chain

Consider a one dimensional harmonic chain, a collection of N point particles, each of mass m, arranged in a line with successive neighbors joined by a spring of spring constant, k arranged

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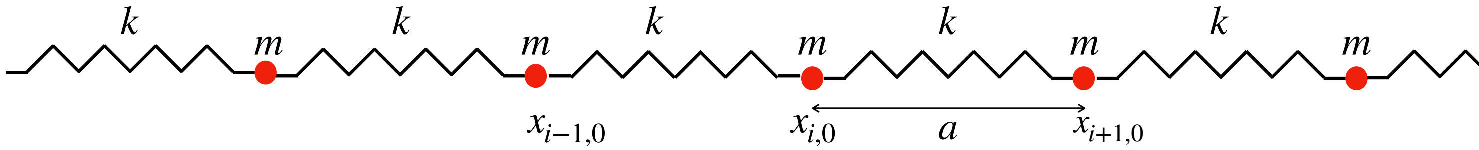

Figure 1: The one dimensional harmonic chain

along the x-axis (see Fig. (1)). Let's say the equilibrium locations of the point masses are, $x_{i,0}$, where the discrete label i, taking values from 1 to N labels the point masses, such that the equilibrium separation(s) is a.

$$x_{i+1,0} - x_{i,0} = a, \forall i \in \{1, \ldots, N\}.$$

In general the position coordinate of the i-th point particle off equilibrium location at a given time t is $x_i(t)$, $i = 1, \ldots, N$. The lagrangian for this system is,

$$L = \sum_{i=1}^{N} \frac{1}{2} m \dot{x}_i^2 - \sum_{i=1}^{N} \frac{1}{2} k \left(x_{i+1} - x_i - a \right)^2.$$

(We have here the sum all kinetic energies of each point mass and the potential energy of each spring). Replacing the a in the i-th element of potential energy sum as,

$$x_{i+1,0} - x_{i,0} = a$$

we get,

$$L = \sum_{i=1}^{N} \frac{1}{2} m \dot{x}_i^2 - \sum_{i=1}^{N} \frac{1}{2} k \left[(x_{i+1} - x_{i+1,0}) - (x_i - x_{i,0}) \right]^2.$$

Next we define, $q_i(t) = x_i(t) - x_{i,0}$ and express the lagrangian in these variables,

$$L = \sum_{i=1}^{N} \frac{1}{2} m \dot{q}_i^2 - \sum_{i=1}^{N} \frac{1}{2} k \left(q_{i+1} - q_i \right)^2. \tag{1}$$

Next we rewrite this as,

$$L = \sum_{i=1}^{N} a \, \frac{1}{2} \frac{m}{a} \, \dot{q}_i^2 - \sum_{i=1}^{N} a \, \frac{1}{2} (ka) \, \left(\frac{q_{i+1} - q_i}{a} \right)^2$$

and take a *continuum limit* by converting this sum into a Riemann sum,

$$a = \Delta x \to 0, \; m \to 0, \; k \to \infty,$$

such that,

$$\mu \equiv \frac{m}{a}, Y \equiv ka \to \text{finite}.$$

Under such a limit the lagrangian becomes,

$$L = \sum_i \Delta x \left[\frac{1}{2}\mu \dot{q}_i^2 - \frac{1}{2} Y \left(\frac{q_{i+1} - q_i}{a} \right)^2 \right]$$

Since in this limit the point particles approach each other i.e. separation vanishes one can label the degrees of freedom q's can be labeled by a continuous variable, namely x, the position coordinate along the x-axis instead of a discrete label, i,

$$q_i \to q(x,t),$$
$$q_{i+1} \to q(x + \Delta x)$$

and hence one can replace,

$$\frac{q_{i+1} - q_i}{a} \to \frac{q(x+\Delta x) - q(x)}{\Delta x} = \frac{\partial q}{\partial x}.$$

(Here I have not shown the time label to reduce cumber in notation but in full rigor one must also include the time label, $q_i(t) \to q(x,t)$ etc.). Finally one can also replace the sum over i's by an integral over x,

$$\sum_i \Delta x \to \int dx.$$

Thus the lagrangian gets transformed into an integral over the (real) line,

$$L = \int dx \left[\frac{1}{2}\mu \left(\frac{\partial q}{\partial t} \right)^2 - \frac{1}{2} Y \left(\frac{\partial q}{\partial x} \right)^2 \right] = \int dx \left(\frac{1}{2}\mu \dot{q}^2 - \frac{1}{2} Y q'^2 \right) \tag{2}$$

i.e. it can be expressed as a line-density,

$$L = \int dx \, \mathcal{L}(\dot{q}, q').$$

This density, $\mathcal{L}(\dot{q}, q') = \frac{1}{2}\mu\dot{q}^2 - \frac{1}{2}Yq'^2$ is called a lagrangian density. From now on we will use the greek symbol $\varphi(x,t)$ instead of $q(x,t)$. In the general case, a lagrangian density is not only a function of the derivatives, $\dot{\varphi}$ and φ' but the field φ itself i.e.

$$\mathcal{L} = \mathcal{L}(\varphi, \dot{\varphi}, \varphi').$$

2.1 Equation of motion for the harmonic chain field theory

The equation of motion for the i-th point mass in the harmonic chain from the lagrangian (1) is,

$$\frac{\partial}{\partial t} \left(\frac{\partial L}{\partial \dot{q}_i} \right) = \frac{\partial L}{\partial q_i},$$

or,

$$m\ddot{q}_i = k(q_{i+1} - q_i) - k\,(q_i - q_{i-1}) \tag{3}$$

or, diving both sides by a,

$$\begin{aligned}\frac{m}{a}\ddot{q}_i &= k\left(\frac{q_{i+1} - q_i}{a}\right) - k\left(\frac{q_i - q_{i-1}}{a}\right)\\ &= ka\,\frac{\left(\frac{q_{i+1}-q_i}{a}\right) - \left(\frac{q_i - q_{i-1}}{a}\right)}{a}.\end{aligned} \tag{4}$$

In the continuum limit, i.e. $a = \Delta x \to 0$, $m = 0$, $k \to \infty$ such that $\mu = \frac{m}{a}$ and $Y = ka$ remain fixed, we can replace

$$\begin{aligned} q_i &\to \varphi(x),\\ q_{i+1} &\to \varphi(x + \Delta x),\\ q_{i-1} &\to \varphi(x - \Delta x).\end{aligned}$$

and the equation of motion (4) becomes,

$$\begin{aligned}\mu\ddot{\varphi} &= Y \lim_{\Delta x \to 0} \frac{\lim_{\Delta x\to 0}\left(\frac{\varphi(x+\Delta x) - \varphi(x)}{\Delta x}\right) - \left(\frac{\varphi(x) - \varphi(x-\Delta x)}{\Delta x}\right)}{\Delta x}\\ &= Y \lim_{\Delta x \to 0} \frac{\frac{\partial \varphi(x+\Delta x)}{\partial x} - \frac{\partial \varphi(x)}{\partial x}}{\Delta x}\\ &= Y\frac{\partial^2 \varphi}{\partial x^2},\end{aligned}$$

or,

$$\varphi'' - \frac{1}{v^2}\ddot{\varphi} = 0,\ v = \sqrt{\frac{Y}{\mu}}. \tag{5}$$

This is of course the one dimensional wave equation with wave velocity, $v = \sqrt{\frac{Y}{\mu}}$.

3 Equation of motion from an action principle for a field theory

The equation of motion for the field theory (5) was obtained in the last section by taking the continuum limit of the Euler-Lagrange equation for the harmonic chain (3). However here we will rederive the equation of motion (5) from Hamilton's principle of least action. Consider a general action for a field theory of a field φ,

$$I[\varphi] = \int_{t_i}^{t_f} dt\, L = \int_{t_i}^{t_f} dt\, dx\, \mathcal{L}(\varphi, \dot{\varphi}, \varphi'), \tag{6}$$

where we have used the fact that for a field theory in 1 space dimensions the Lagrangian, L is a line-density, i.e. $L = \int dx\, \mathcal{L}$ for a function $\mathcal{L}$ of the field and its first derivatives,

$$\mathcal{L} = \mathcal{L}(\varphi, \dot{\varphi}, \varphi')$$

According to Hamilton's principle, one has to vary the field,

$$\varphi(x,t) \to \varphi(x,t) + \delta\varphi(x,t),$$

with the "end-point" values of the field fixed, i.e. $\delta\varphi(x, t_i) = \delta\varphi(x, t_f) = 0$ for all x, and demand that the first order change in the action (as a result of varying the field) vanishes,

$$\delta I[\varphi] = I[\varphi(x,t) + \delta\varphi(x,t)] - I[\varphi(x,t)] = 0,$$

for ***arbitrary variation of the field*** $\delta\varphi$. This vanishing first order change in the action condition (stationary action/ least action/ extremal action) then leads to the equation of motion for the field, φ.

Let's apply Hamilton's principle for a general one (space) dimensional field theory lagrangian (6) to derive the equation of motion. The new action after changing the field, i.e. when we replace $\varphi(x,t) \to \varphi(x,t) + \delta\varphi(x,t)$, is

$$I[\varphi + \delta\varphi] = \int_{t_i}^{t_f} dt\, dx\, \mathcal{L}(\varphi + \delta\varphi, \dot{\varphi} + \delta\dot{\varphi}, \varphi' + \delta\varphi')$$

Since the lagrangian density is a function of the field as well as its space and time derivatives, we expand the lagrangian density to first order in each of the variables on which it depends,

$$\mathcal{L}(\varphi + \delta\varphi, \dot{\varphi} + \delta\dot{\varphi}, \varphi' + \delta\varphi') = \mathcal{L}(\varphi, \dot{\varphi}, \varphi') + \frac{\partial \mathcal{L}}{\partial \varphi}\delta\varphi + \frac{\partial \mathcal{L}}{\partial \dot{\varphi}}\delta\dot{\varphi} + \frac{\partial \mathcal{L}}{\partial \varphi'}\delta\varphi' + O((\delta\varphi)^2, (\delta\dot{\varphi})^2, (\delta\varphi')^2).$$

Hence the changed action to first order in variation of fields and its derivatives is,

$$\begin{aligned} I[\varphi + \delta\varphi] &= \int_{t_i}^{t_f} dt\, dx\, \mathcal{L}(\varphi + \delta\varphi, \dot{\varphi} + \delta\dot{\varphi}, \varphi' + \delta\varphi') \\ &= \underbrace{\int_{t_i}^{t_f} dt\, dx\, \mathcal{L}(\varphi, \dot{\varphi}, \varphi')}_{=I[\varphi]} + \int_{t_i}^{t_f} dt\, dx\, \left(\frac{\partial \mathcal{L}}{\partial \varphi}\delta\varphi + \frac{\partial \mathcal{L}}{\partial \dot{\varphi}}\delta\dot{\varphi} + \frac{\partial \mathcal{L}}{\partial \varphi'}\delta\varphi' + O((\delta\varphi)^2, (\delta\dot{\varphi})^2, (\delta\varphi')^2)\right). \end{aligned}$$

Thus the first order change is,

$$\delta I[\varphi] = I[\varphi + \delta\varphi] - I[\varphi] = \int_{t_i}^{t_f} dt\, dx\, \left(\frac{\partial \mathcal{L}}{\partial \varphi}\delta\varphi + \frac{\partial \mathcal{L}}{\partial \dot{\varphi}}\delta\dot{\varphi} + \frac{\partial \mathcal{L}}{\partial \varphi'}\delta\varphi'\right).$$

Next we commute the variation and derivative order, i.e. $\delta\dot{\varphi} = (\dot{\delta\varphi})$and $\delta\varphi' = (\delta\phi)'$ and get,

$$\delta I[\varphi] = \int_{t_i}^{t_f} dt\, dx\, \left(\frac{\partial \mathcal{L}}{\partial \varphi}\delta\varphi + \frac{\partial \mathcal{L}}{\partial \dot{\varphi}}(\dot{\delta\varphi}) + \frac{\partial \mathcal{L}}{\partial \varphi'}(\delta\phi)'\right). \qquad (7)$$

Next we use the product rule of differentiation to write,

$$\frac{\partial \mathcal{L}}{\partial \dot{\varphi}}(\dot{\delta\varphi}) = \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\varphi}}\delta\varphi\right) - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\varphi}}\right)\delta\varphi,$$

$$\frac{\partial \mathcal{L}}{\partial \varphi'}(\delta\phi)' = \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial \varphi'}\delta\varphi\right) - \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial \varphi'}\right)\delta\varphi$$

and substitute in the first order change of the action (7) to obtain,

$$\delta I[\varphi] = \int_{t_i}^{t_f} dtdx\left[\frac{\partial \mathcal{L}}{\partial \varphi} - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\varphi}}\right) - \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial \varphi'}\right)\right]\delta\varphi + \int dx \int_{t_i}^{t_f} dt\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\varphi}}\delta\varphi\right) + \int_{t_i}^{t_f} dt \int dx \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial \varphi'}\delta\varphi\right)$$

The last two terms in the RHS are total time or space derivative terms and hence can be integrated over time and space respectively, which yields

$$\begin{aligned}\delta I[\varphi] = \int_{t_i}^{t_f} dt\, dx\, &\left[\frac{\partial \mathcal{L}}{\partial \varphi} - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\varphi}}\right) - \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial \varphi'}\right)\right]\delta\varphi \\ &+ \int dx\left(\frac{\partial \mathcal{L}}{\partial \dot{\varphi}}\delta\varphi(x,t_f) - \frac{\partial \mathcal{L}}{\partial \dot{\varphi}}\delta\varphi(x,t_i)\right) + \int_{t_i}^{t_f} dt\left(\frac{\partial \mathcal{L}}{\partial \varphi'}\delta\varphi\Big|_{x_R} - \frac{\partial \mathcal{L}}{\partial \varphi'}\delta\varphi\Big|_{x_L}\right)\end{aligned}$$

Since in Hamilton's principle, the initial and final configurations are fixed (fixed endpoints), i.e. $\delta\varphi(x,t_f) = \delta\varphi(x,t_i) = 0$, the middle term vanishes and the first order variation of the action becomes,

$$\begin{aligned}\delta I[\varphi] = \int_{t_i}^{t_f} dt\, dx\, &\left[\frac{\partial \mathcal{L}}{\partial \varphi} - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\varphi}}\right) - \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial \varphi'}\right)\right]\delta\varphi \\ &+ \int_{t_i}^{t_f} dt\left(\frac{\partial \mathcal{L}}{\partial \varphi'}\delta\varphi\Big|_{x_R} - \frac{\partial \mathcal{L}}{\partial \varphi'}\delta\varphi\Big|_{x_L}\right).\end{aligned} \tag{8}$$

Now in order to well-define the variational principle one also needs to get rid of the second surface or boundary term, i.e. which has support at the left and right spatial boundary x_L and x_R. One can accomplish this by imposing the following boundary conditions,

1. *Periodic boundary conditions*: Here x_L and x_R are identified to be the same point (circular chain), and as a result,

   $$\frac{\partial \mathcal{L}}{\partial \varphi'}\delta\varphi\Big|_{x_R} = \frac{\partial \mathcal{L}}{\partial \varphi'}\delta\varphi\Big|_{x_R} \;\forall\, t$$

   and the boundary term being proportional to the difference vanishes.

2. *Dirichlet boundary conditions*: Here we set the boundary variations of the fields to zero i.e. fixed boundary conditions (just like fixed initial and final conditions),

   $$\delta\varphi(x_L,t) = 0 = \delta\varphi(x_R,t), \forall\, t$$

   As a result the boundary term also vanishes.

3. *Neumann boundary conditions*: Here we keep allow the boundary variation, $\delta\varphi(x_L/x_R)$, to be arbitrary but instead impose at the boundary

$$\left.\frac{\partial \mathcal{L}}{\partial \varphi'}\right|_{x_R} = 0 = \left.\frac{\partial \mathcal{L}}{\partial \varphi'}\right|_{x_L}$$

   This also makes the boundary term vanish.

4. *Mixed Dirichlet and Neumann boundary conditions*: Here we can impose separate (independent) boundary conditions for the left and right spatial boundaries, e.g., Neumann for left boundary and Dirichlet for right boundary or vice versa.

Once the spatial boundary (surface) term in the first order change in the action expression (8) is made to vanish via suitable boundary conditions listed above the action principle gets well defined,

$$\delta I[\varphi] = \int_{t_i}^{t_f} dt\, dx \left[\frac{\partial \mathcal{L}}{\partial \varphi} - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\varphi}}\right) - \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial \varphi'}\right)\right] \delta\varphi.$$

Now ala Hamilton we can demand this first order change to vanish, i.e. $I[\delta\varphi] = 0$ for **arbitrary** variations $\delta\varphi$. It is evident that this integral will vanish for arbitrary $\delta\varphi$ if and only if, the rest of the integrand vanishes, i.e.,

$$\frac{\partial \mathcal{L}}{\partial \varphi} - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\varphi}}\right) - \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial \varphi'}\right) = 0. \tag{9}$$

This is the equation of motion of a (1 space dimensional) field theory.

As an example we check whether the general equation of motion (9) indeed reproduces the wave equation, (5), for the field theory which resulted from the continuum limit of the harmonic chain . For this case, $\mathcal{L} = \frac{\mu}{2}\dot{\varphi}^2 - \frac{Y}{2}\varphi'^2$, and we have,

$$\frac{\partial \mathcal{L}}{\partial \varphi} = 0, \qquad \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = \mu\dot{\varphi}, \qquad \frac{\partial \mathcal{L}}{\partial \varphi'} = -Y\varphi'.$$

Plugging these in (9), we get,

$$\begin{aligned} 0 - \frac{\partial}{\partial t}(\mu\dot{\varphi}) - \frac{\partial}{\partial x}(-Y\varphi') &= 0, \\ \Rightarrow \mu\ddot{\varphi} - Y\varphi'' &= 0, \end{aligned}$$

which is indeed same as (5).

Comments:

- It is easy to extend the math to arbitrary number of spatial dimensions. One just needs to include as many terms to the equation of motion (9), as there are number of spacetime dimensions. E.g. in 3 space and 1 time dimensions the EOM will look like,

$$\frac{\partial \mathcal{L}}{\partial \varphi} - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\varphi}}\right) - \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial \varphi_{,x}}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \mathcal{L}}{\partial \varphi_{,y}}\right) - \frac{\partial}{\partial z}\left(\frac{\partial \mathcal{L}}{\partial \varphi_{,z}}\right) = 0$$

  where $\varphi_{,x} \equiv \frac{\partial \varphi}{\partial x}$, $\varphi_{,y} \equiv \frac{\partial \varphi}{\partial y}$, and $\varphi_{,z} = \frac{\partial \varphi}{\partial z}$. Using vector notation one can express this Euler-Lagrange equation more compactly

$$\frac{\partial \mathcal{L}}{\partial \varphi} - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\varphi}}\right) - \boldsymbol{\nabla} \cdot \left(\frac{\partial \mathcal{L}}{\partial \left(\boldsymbol{\nabla} \varphi\right)}\right) = 0.$$

  In fact, one can make the equation of motion appear more compact by using relativistic 4-vector notation,

$$\frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \varphi\right)}\right) = 0,$$

  where ∂_μ is the covariant 4-component derivative operator, $\partial_\mu = \left(\frac{1}{c}\partial_t, \boldsymbol{\nabla}\right)$. This is the form of the Euler-Lagrange equation we will use when discussing fields theories with Lorentz symmetry (relativistic field theories).

- The variational principle (extremum action principle) roves to be a very powerful tool as it can determine both the equations of motion of a system, as well as the allowed boundary conditions.

- We see that the first order change (variation) in the action leads to the equation of motion of a physical system. What about the second order change? What is its role in physics? (Hint: Quantum fluctuations).

References

[1] Goldstein, Poole and Safko, "Classical Mechanics", Sec. 13.1-13.2

[2] Jose and Saletan, "Classical Dynamics: A contemporary approach", Sec 9.1

[3] Sakurai, "Advanced Quantum Mechanics", Sec 1.2