Quantum Field Theory: PH6418/ EP4618 (Fall 2022)

Notes for Lecture 2*

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1 Functionals

The branch of mathematics which is custom-made for treating field theories in physics is functional calculus. This is a generalization of multivariable calculus i.e. calculus of a discrete finite number of variables to the case when the number of variables become infinite (i.e. a field). First we need to define what a functional is. To this end we recall that in calculus, a function, $f(x)$ of a real variable x is defined to be a map from $\mathbb{R}$ to $\mathbb{R}$,

$$f : \mathbb{R} \to \mathbb{R},$$

or a map from $\mathbb{C}$ to $\mathbb{C}$ for a function $f(z)$ of a complex variable, z.

A functional is defined to be a map from the set of real (or complex) function(s) to the $\mathbb{R}$ (or $\mathbb{C}$ if it is a complex functional). Let's say if $f(x)$ is a function of a single variable, then a functional of $f(x)$, denote it by $F[f]$ or $F[f(x)]$ is map from all possible functions, $f(x)$ to real numbers:

$$F : \{f(x)\} \to \mathbb{R}.$$

An example of a functional is $F[f(x)] = \int_0^1 dx\ f(x)$. As is evident from Table 1, the functional $F[f]$ inputs a function from the left column and outputs a real number.

Just as one can generalize from a function of a single real (complex) variable to a function of several real (complex) variables say $f(x_1, x_2, \ldots, x_N)$, one can easily generalize this concept to a functional of several real (complex) functions, say $F[f_1(x), f_2(x), f_3(x), \ldots f_N(x)]$, which is a map from the "space of N real (complex) functions" to the real line $\mathbb{R}$ (or the complex plane, $\mathbb{C}$).

Although one necessarily needs to introduce functionals for field theory, the fact is that functionals can be used even for discrete number of degrees of freedom, e.g. a single particle as we will

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$f(x)$	$F[f(x)]$
$x+2x^2$	$1.166\ldots$
e^x	$1.718\ldots$
$\sin x$	0.459...
$\ln x$	-1

Table 1: $F[f]$ is a functional of the function $f(x)$

see in the following. However for discrete degrees of freedom using functional calculus is overkill, so we usually omit discussing functionals in the context of particle classical mechanics. Consider the action for a non-relativistic free point particle of mass m (for simplicity moving in one space dimension), described by the well known action,

$$I_{PP}[q(t)] = \int dt \, \left(\frac{m}{2}\,\dot{q}^2\right) \tag{1}$$

Clearly, the action is a (real) functional of the trajectory function $q(t)$ - for every trajectory function, $q(t)$ the action integral generates some real number. If the particle is allowed to move in three space dimensions the action becomes a functional of three real functions representing the trajectory, $x(t)$, $y(t)$ and $z(t)$, namely,

$$I[x(t), y(t), z(t)] = \int dt \, \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right).$$

or using 3-vector notation,

$$I[\boldsymbol{x}(t)] = \int dt \, \left(\frac{m}{2}\dot{\boldsymbol{x}}^2\right).$$

Comment:

Can we regard a functional as a "function of a function"? The answer is, yes in some ways, but it is not a good way to think about functionals because usually in math a function of a function is something which is known as a *composite map* or *composite function* e.g.for two functions $f(x)$ and $g(x)$ one can define composite maps (function of a function), $f \circ g(x) \equiv f(g(x))$ and $g \circ f(x) \equiv g(f(x))$ which are new functions but not functionals.

1.1 Functional Differentiation

Now the next thing one learns in usual multivariable calculus after defining functions is the concept of a derivative of a function. Analogously one can define in functional calculus, the ***functional derivative*** , namely,

$$\frac{\delta F[f]}{\delta f(x)}$$

which measures how the functional, F responds to changes in the function, $f(x)$ on which it depends. We will learn how to take (define) functional derivatives of a functional from analogy with multivariable calculus and then passing over to continuum limit i.e.infinite variables or a field. First note if we have N real variables, q_i, where $i = 1, \ldots, N$, one has the rule (definition),

$$\frac{\partial q_i}{\partial q_j} = \delta_{ij}. \tag{2}$$

Now when the number of variables becomes infinite and get labeled by a continuous parameter, say x instead of discrete index label, i, we have to replace

$$q_i \to \varphi(x)$$

and also,

$$q_j \to \varphi(y).$$

Then the partial derivative on the LHS of (2) becomes converted to the functional derivative,

$$\frac{\partial q_i}{\partial q_j} \to \frac{\delta\varphi(x)}{\delta\varphi(y)}.$$

As one can naturally guess the Kronecker delta, δ_{ij} on the RHS of (2) will get converted to a Dirac delta, $\delta(x-y)$! Thus we have the first rule for functional derivatives,

$$\frac{\delta\varphi(x)}{\delta\varphi(y)} = \delta(x-y) \tag{3}$$

The second rule is the Leibniz rule for differentiation, namely, functional derivative of a product of several quantities splits up into a sum,

$$\frac{\delta}{\delta\varphi(x)}(A\,B\,C\ldots) = \frac{\delta A}{\delta\varphi(x)}\,B\,C\ldots + A\,\frac{\delta B}{\delta\varphi(x)}\,C\ldots + A\,B\,\frac{\delta C}{\delta\varphi(x)}\ldots + \ldots. \tag{4}$$

The third rule is the analog of the chain rule of multivariable calculus, namely,

$$\frac{\partial f(q_i)}{\partial q_j} = \frac{\partial f(q_i)}{\partial q_i}\,\frac{\partial q_i}{\partial q_j},$$

We have a similar rule for functional derivatives of a *function*

$$\frac{\delta f\,(\varphi(x))}{\delta\varphi(y)} = \frac{\partial f\,(\varphi(x))}{\partial\varphi(x)}\frac{\delta\varphi(x)}{\delta\varphi(y)} \tag{5}$$

Be careful about this chain rule since *it contains both partial derivatives and functional derivatives on the RHS!*.

The fourth and final rule we will need to take functional derivatives is the (perhaps obvious) one where we state that functional derivatives commute with normal partial spacetime derivatives and space or time integrals:

$$
\begin{aligned}
\frac{\delta}{\delta\varphi(y)}\,\frac{\partial}{\partial x}\,(\ldots) &= \frac{\partial}{\partial x}\,\frac{\delta}{\delta\varphi(x)}\,(\ldots)\,, \\
\frac{\delta}{\delta\varphi(y)}\int dt\,dx\,(\ldots) &= \int dt\,dx\,\frac{\delta}{\delta\varphi(y)}\,(\ldots)\,.
\end{aligned}
\tag{6}
$$

1.2 Functional Euler-Lagrange equations for a field theory

Here we generalize the Euler-Lagrange (EL) equations for a physical system with discrete number of degrees of freedom, say q_i, $i = 1, \ldots, N$ to field theory. For discrete number of degrees of freedom, the EL equations read,

$$
\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i} \qquad \forall i = 1, \ldots, N.
$$

To write down the analogous equations for field theory we make the following replacement (the discrete index i to a continuous label, x),

$$
\begin{aligned}
q_i &\to \varphi(x), \\
\dot{q}_i &\to \dot{\varphi}(x),
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial}{\partial q_i} &\to \frac{\delta}{\delta\varphi(x)}, \\
\frac{\partial}{\partial \dot{q}_i} &\to \frac{\delta}{\delta\dot{\varphi}(x)},
\end{aligned}
$$

and get,

$$
\frac{\partial}{\partial t}\left(\frac{\delta L}{\delta\dot{\varphi}(x)}\right) = \frac{\delta L}{\delta\varphi(x)}, \qquad \forall x.
\tag{7}
$$

Comments:

Throughout this subsection we have suppressed the time-dependence of the generalized coordinates, i.e. we have used q_i instead of $q_i(t)$, and analogously suppressed the time-dependence of the field φ, i.e. we have used the notation $\varphi(x)$ instead $\varphi(x, t)$. This has been done to reduce the clutter of notation and no information is lost while suppressing the time-dependence in this notation. This is what is traditionally done even in classical mechanics of a point particle (or mechanics of a single d.o.f.), where the equation of motion is stated in the form,

$$\frac{\partial L}{\partial q} = \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{q}}\right),$$

instead of the more cumbersome

$$\frac{\partial L}{\partial q(t)} = \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{q}(t)}\right).$$

1.3 Check: Reproducing the EOM for the harmonic chain field theory using the functional EL equations

The field theory lagrangian in this case is,

$$L = \int dx \, \left(\frac{\mu}{2}\dot{\varphi}^2(x) - \frac{Y}{2}\left(\varphi'(x)\right)^2\right). \tag{8}$$

Then, we have,

$$\begin{aligned}
\frac{\delta L}{\delta \dot{\varphi}(y)} &= \frac{\delta}{\delta \dot{\varphi}(y)} \int dx \, \left(\frac{\mu}{2}\dot{\varphi}^2(x) - \frac{Y}{2}\left(\varphi'(x)\right)^2\right) \\
&= \int dx \, \frac{\delta}{\delta \dot{\varphi}(y)} \, \left(\frac{\mu}{2}\dot{\varphi}^2(x) - \frac{Y}{2}\left(\varphi'(x)\right)^2\right) \\
&= \int dx \, \left(\frac{\mu}{2}\frac{\delta \dot{\varphi}^2(x)}{\delta \dot{\varphi}(y)} - \frac{Y}{2}\cancelto{0}{\frac{\delta}{\delta \dot{\varphi}(y)}\left(\varphi'(x)\right)^2}\right) \\
&= \int dx \, \frac{\mu}{2} \, 2\dot{\varphi}(x) \, \underbrace{\frac{\delta \dot{\varphi}(x)}{\delta \dot{\varphi}(y)}}_{=\delta(x-y)} \\
&= \mu \int dx \, \dot{\varphi}(x) \, \delta(x-y) \\
&= \mu \, \dot{\varphi}(y).
\end{aligned} \tag{9}$$

So the LHS of the EL equation (7) is,

$$\frac{\partial}{\partial t}\left(\frac{\delta L}{\delta \dot{\varphi}(y)}\right) = \frac{\partial}{\partial t}\left(\mu \, \dot{\varphi}(y)\right) = \mu \, \ddot{\varphi}(y) \tag{10}$$

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This is where the lecture ended. This rest of the ssubection will be covered in lec.. 3

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Next we compute the RHS of the EL equation (7)

$$
\begin{aligned}
\frac{\delta L}{\delta\varphi(y)} &= \frac{\delta}{\delta\varphi(y)}\int dx\ \left(\frac{\mu}{2}\dot{\varphi}^2(x) - \frac{Y}{2}\left(\varphi'(x)\right)^2\right) \\
&= \int dx\ \frac{\delta}{\delta\varphi(y)}\left(\frac{\mu}{2}\dot{\varphi}^2(x) - \frac{Y}{2}\left(\varphi'(x)\right)^2\right) \\
&= \int dx\ \left(\frac{\mu}{2}\cancelto{0}{\frac{\delta\dot{\varphi}^2(x)}{\delta\varphi(y)}} - \frac{Y}{2}\frac{\delta\left(\varphi'(x)\right)^2}{\delta\varphi(y)}\right) \\
&= \int dx\left(-\frac{Y}{2}\right) 2\,\varphi'(x)\,\frac{\delta\varphi'(x)}{\delta\varphi(y)} \\
&= -Y\int dx\ \varphi'(x)\ \frac{\delta}{\delta\varphi(y)}\ \frac{\partial}{\partial x}\varphi(x) \\
&= -Y\int dx\ \varphi'(x)\ \frac{\partial}{\partial x}\ \underbrace{\frac{\delta}{\delta\varphi(y)}\varphi(x)}_{=\delta(x-y)} \\
&= -Y\int dx\ \varphi'(x)\ \frac{\partial}{\partial x}\delta(x-y) \\
&= Y\int dx\ \frac{\partial}{\partial x}\varphi'(x)\ \delta(x-y) \\
&= Y\frac{\partial^2\varphi(y)}{\partial y^2}.
\end{aligned}
\tag{11}
$$

Now that we have computed both sides of the EL equation (7) for this, we can just write it down,

$$
\begin{aligned}
\mu\ddot{\varphi} &= Y\varphi'', \\
\Rightarrow \varphi'' - \frac{1}{v^2}\ddot{\varphi} &= 0, \quad v = \sqrt{\frac{Y}{\mu}}.
\end{aligned}
\tag{12}
$$

This familiar equation of motion is of course what we expected to get.

Comments:

- Observe that we have take the lagrangian (density) to be a function of the field and its first derivative, i.e. $\dot{\varphi}, \varphi'$. Can we take a more general lagrangian which depends not only on the field and its first space-time derivatives but higher order spacetime derivatives? The answer is if one considers more general lagrangian which depend on higher order time derivatives of the field, the physical system develops instabilities, a phenomenon which was uncovered by

M. Ostrogradsky and goes by his name, the *Ostrogradsky instability*.

- For the sake of simplicity we have discussed a field theory with one space dimension. However our entire discussion can be straightforwardly generalized to three space dimensions (or arbitrary d dimensions). In that case the Lagrangian density will be a function of all spatial derivatives,

$$\mathcal{L} = \mathcal{L}(\varphi, \dot{\varphi}, \boldsymbol{\nabla}\varphi),$$

and the lagrangian will be volume integral of the density,

$$L = \int d^3\boldsymbol{x}\ \mathcal{L}(\varphi, \dot{\varphi}, \boldsymbol{\nabla}\varphi).$$

For example one can easily generalize the harmonic chain field theory to three dimensions,

$$\mathcal{L} = \frac{\rho}{2}\dot{\varphi}^2 - \frac{B}{2}\left(\boldsymbol{\nabla}\varphi\right) \cdot \left(\boldsymbol{\nabla}\varphi\right)$$

where ρ is now the volume mass-density and B is the bulk modulus.

References

[1] Ashok Das, "Field theory: A path integral approach", Sec 1.3 (for functionals)