Quantum Field Theory: PH6418/EP4618 (Spring 2022)

Lecture 3 notes*

February 24, 2022

1 Functionals II: Functional Euler-Lagrange equations for a field theory

Here we generalize the Euler-Lagrange (EL) equations for a physical system with discrete number of degrees of freedom, say q_i, $i = 1, \ldots, N$ to field theory. For discrete number of degrees of freedom, the EL equations read,

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i} \qquad \forall i = 1, \ldots, N.$$

To write down the analogous equations for field theory we make the following replacement (the discrete index i to a continuous label, x),

$$q_i \to \varphi(x),$$
$$\dot{q}_i \to \dot{\varphi}(x),$$

and

$$\frac{\partial}{\partial q_i} \to \frac{\delta}{\delta \varphi(x)},$$
$$\frac{\partial}{\partial \dot{q}_i} \to \frac{\delta}{\delta \dot{\varphi}(x)},$$

and get,

$$\frac{\partial}{\partial t}\left(\frac{\delta L}{\delta \dot{\varphi}(x)}\right) = \frac{\delta L}{\delta \varphi(x)}, \qquad \forall x. \tag{1}$$

1.1 Check: Reproducing the EOM for the harmonic chain field theory using the functional EL equations

The field theory lagrangian in this case is,

$$L = \int dx \left(\frac{\mu}{2}\dot{\varphi}^2(x) - \frac{Y}{2}\left(\varphi'(x)\right)^2\right). \tag{2}$$

*Typos and errors should be emailed to the Instructor: Shubho Roy (email: sroy@phy.iith.ac.in)

Then, we have,

$$
\begin{aligned}
\frac{\delta L}{\delta\dot\varphi(y)} &= \frac{\delta}{\delta\dot\varphi(y)}\int dx\ \left(\frac{\mu}{2}\dot\varphi^2(x) - \frac{Y}{2}\left(\varphi'(x)\right)^2\right)\\
&= \int dx\ \frac{\delta}{\delta\dot\varphi(y)}\ \left(\frac{\mu}{2}\dot\varphi^2(x) - \frac{Y}{2}\left(\varphi'(x)\right)^2\right)\\
&= \int dx\ \left(\frac{\mu}{2}\frac{\delta\dot\varphi^2(x)}{\delta\dot\varphi(y)} - \frac{Y}{2}\cancelto{0}{\frac{\delta}{\delta\dot\varphi(y)}\ \left(\varphi'(x)\right)^2}\right)\\
&= \int dx\ \frac{\mu}{2}\ 2\dot\varphi(x)\ \underbrace{\frac{\delta\dot\varphi(x)}{\delta\dot\varphi(y)}}_{=\delta(x-y)}\\
&= \mu\int dx\ \dot\varphi(x)\ \delta(x-y)\\
&= \mu\ \dot\varphi(y).
\end{aligned}
\tag{3}
$$

So the LHS of the EL equation (1) is,

$$
\frac{\partial}{\partial t}\left(\frac{\delta L}{\delta\dot\varphi(y)}\right) = \frac{\partial}{\partial t}\left(\mu\ \dot\varphi(y)\right) = \mu\ \ddot\varphi(y)
\tag{4}
$$

Next we compute the RHS of the EL equation (1)

$$
\begin{aligned}
\frac{\delta L}{\delta\varphi(y)} &= \frac{\delta}{\delta\varphi(y)}\int dx\ \left(\frac{\mu}{2}\dot\varphi^2(x) - \frac{Y}{2}\left(\varphi'(x)\right)^2\right)\\
&= \int dx\ \frac{\delta}{\delta\varphi(y)}\left(\frac{\mu}{2}\dot\varphi^2(x) - \frac{Y}{2}\left(\varphi'(x)\right)^2\right)\\
&= \int dx\ \left(\frac{\mu}{2}\cancelto{0}{\frac{\delta\dot\varphi^2(x)}{\delta\varphi(y)}} - \frac{Y}{2}\frac{\delta\left(\varphi'(x)\right)^2}{\delta\varphi(y)}\right)\\
&= \int dx\left(-\frac{Y}{2}\right)2\ \varphi'(x)\ \frac{\delta\varphi'(x)}{\delta\varphi(y)}\\
&= -Y\int dx\ \varphi'(x)\ \frac{\delta}{\delta\varphi(y)}\ \frac{\partial}{\partial x}\varphi(x)\\
&= -Y\int dx\ \varphi'(x)\ \frac{\partial}{\partial x}\ \underbrace{\frac{\delta}{\delta\varphi(y)}\varphi(x)}_{=\delta(x-y)}\\
&= -Y\int dx\ \varphi'(x)\ \frac{\partial}{\partial x}\delta(x-y)\\
&= Y\int dx\ \frac{\partial}{\partial x}\varphi'(x)\ \delta(x-y)\\
&= Y\frac{\partial^2\varphi(y)}{\partial y^2}.
\end{aligned}
\tag{5}
$$

Now that we have computed both sides of the EL equation (1) for this, we can just write it down,

$$\mu \ddot{\varphi} = Y \varphi'',$$
$$\Rightarrow \varphi'' - \frac{1}{v^2}\ddot{\varphi} = 0, \quad v = \sqrt{\frac{Y}{\mu}}. \tag{6}$$

This familiar equation of motion is of course what we expected to get.

Comments:

- Observe that we have take the lagrangian (density) to be a function of the field and its first derivative, i.e. $\dot{\varphi}, \varphi'$. Can we take a more general lagrangian which depends not only on the field and its first space-time derivatives but higher order spacetime derivatives? The answer is if one considers more general lagrangian which depend on higher order time derivatives of the field, the physical system develops instabilities, a phenomenon which was uncovered by M. Ostrogradsky and goes by his name, the *Ostrogradsky instability.*

- For the sake of simplicity we have discussed a field theory with one space dimension. However our entire discussion can be straightforwardly generalized to three space dimensions (or arbitrary d dimensions). In that case the Lagrangian density will be a function of all spatial derivatives,

  $$\mathcal{L} = \mathcal{L}(\varphi, \dot{\varphi}, \boldsymbol{\nabla}\varphi),$$

  and the lagrangian will be volume integral of the density,

  $$L = \int d^3\boldsymbol{x}\ \mathcal{L}(\varphi, \dot{\varphi}, \boldsymbol{\nabla}\varphi).$$

  For example one can easily generalize the harmonic chain field theory to three dimensions,

  $$\mathcal{L} = \frac{\rho}{2}\dot{\varphi}^2 - \frac{B}{2}\left(\boldsymbol{\nabla}\varphi\right)\cdot\left(\boldsymbol{\nabla}\varphi\right)$$

  where ρ is now the volume mass-density and B is the bulk modulus.

- For a generic lagrangian,

  $$L = \int dx\ \mathcal{L}(\varphi, \varphi', \dot{\varphi})$$

  the functional Euler-Lagrange equation (1), after evaluating the functional derivatives, read as

  $$\frac{\partial \mathcal{L}}{\partial \varphi} - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\varphi}}\right) - \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial \varphi'}\right) = 0, \tag{7}$$

  i.e. the same form as the one we derived by varying the action.

2 Hamiltonian description of field theory: Canonical momentum, Hamiltonian (density) and Hamilton's equations

For the discrete case, the generalized momentum is defined as,

$$\pi_i = \frac{\partial L}{\partial \dot{q}_i}$$

To obtain the analogous quantity for the continuum case, replace the discrete index, i, by continuum label x,

$$i \to x,$$
$$\dot{q} \to \dot{\varphi}(x),$$

and as a result we need to replace the partial derivative by functional derivative,

$$\frac{\partial}{\partial \dot{q}_i} \to \frac{\delta}{\delta \dot{\varphi}(x)}.$$

Thus we have the canonical momentum for the continuum case,

$$\pi(x) = \frac{\delta L}{\delta \dot{\varphi}(x)}. \tag{8}$$

Since a field theory Lagrangian is the integral of a density, i.e. $L = \int dx\, \mathcal{L}(\varphi, \dot{\varphi}, \varphi')$, one can check that working out the functional derivative using chain rule of functional differentiation that,

$$\pi(x) = \frac{\delta L}{\delta \dot{\varphi}(x)} = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}(x)}. \tag{9}$$

This gives a nice expression for the momentum in a field theory in terms of partial derivatives (instead of functional derivatives).

The Hamiltonian for a discrete system is defined via the multivariable Legendre transform,

$$H = \sum_i p_i\, \dot{q}_i - L.$$

To obtain the corresponding expression for the field theory (continuum system) we replace,

$$\sum_i \to \int dx,$$
$$p_i \to \pi(x),$$
$$\dot{q}_i \to \dot{\varphi}(x),$$

and thus we have,

$$H = \int dx\, \pi(x)\, \dot{\varphi}(x) - L, \tag{10}$$

but since for a field theory the lagrangian itself is the integral of a density, $L = \int dx\, \mathcal{L}$, we have the Hamiltonian to be an integral of a density as well,

$$H = \int dx\, \mathcal{H}, \tag{11}$$

$$\mathcal{H}\left(\pi(x), \varphi(x)\right) = \pi(x)\, \dot{\varphi}(x) - \mathcal{L}\left(\varphi(x), \dot{\varphi}(x), \varphi'(x)\right), \tag{12}$$

where, as usual, in the final expression one has to invert (9) to express the generalized velocity in terms of momentum and field, $\dot{\varphi}(x) = \dot{\varphi}\left(\pi(x), \varphi(x), \varphi'(x)\right)$.

On applying the variational principle to the action (this is the pq form of the action)

$$I = \int dt\, [p_i \dot{q}_i - H(p_i, q_i)]$$

one can obtain Hamilton's equations of evolution for a system of several discrete degrees of freedom:

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}.$$

The Hamilton's equations can be immediately generalized to continuum case (field theory), by replacing all discrete indices, i by the continuum label, x,

$$q_i \to \varphi(x), \quad p_i \to \pi(x)$$

as well as replacing the partial derivatives by functional derivatives,

$$\frac{\partial}{\partial q_i} \to \frac{\delta}{\delta\varphi(x)}, \qquad \frac{\partial}{\partial p_i} \to \frac{\delta}{\delta\pi(x)}.$$

The Hamilton's equations are,

$$\dot{\varphi}(x) = \frac{\delta H}{\delta\pi(x)}, \qquad \dot{\pi}(x) = -\frac{\delta H}{\delta\varphi(x)} \tag{13}$$

Using chain rule for functional differentiation for the RHS of the Hamilton's equations (13), we get the Hamilton's equations involving just partial derivatives, instead of functional derivatives,

$$\dot{\varphi}(x) = \frac{\partial\mathcal{H}}{\partial\pi(x)}, \qquad \dot{\pi}(x) = -\frac{\partial\mathcal{H}}{\partial\varphi(x)} + \frac{\partial}{\partial x}\left(\frac{\partial\mathcal{H}}{\partial\varphi'(x)}\right). \tag{14}$$

Comments:

- For the case of three space dimensions one just need to replace $x \to \boldsymbol{x}$, and $\frac{\partial}{\partial x} \to \boldsymbol{\nabla}$ in all equations, e.g.

$$\dot{\varphi}(\boldsymbol{x}) = \frac{\partial\mathcal{H}}{\partial\pi(\boldsymbol{x})}, \qquad \dot{\pi}(\boldsymbol{x}) = -\frac{\partial\mathcal{H}}{\partial\varphi(\boldsymbol{x})} + \boldsymbol{\nabla}\cdot\left(\frac{\partial\mathcal{H}}{\partial\left(\boldsymbol{\nabla}\varphi(x)\right)}\right).$$

2.1 Example: Momentum, Hamiltonian (density) and Hamilton's equations for harmonic chain field theory

Consider the field theory given the lagrangian (2). The momentum for this system (using (3)) is,

$$\pi(x) = \frac{\delta L}{\delta \dot{\varphi}(x)} = \mu \; \dot{\varphi}(x).$$

This gives,

$$\dot{\varphi}(x) = \frac{\pi(x)}{\mu}.$$

The Hamiltonian density is then

$$\begin{aligned}
\mathcal{H} &= \pi \; \dot{\varphi} - \mathcal{L} \\
&= \pi \; \dot{\varphi} - \left(\frac{\mu}{2} \dot{\varphi}^2 - \frac{Y}{2} \varphi'^2 \right) \\
&= \pi \; \frac{\pi}{\mu} - \left(\frac{\mu}{2} \left(\frac{\pi}{\mu} \right)^2 - \frac{Y}{2} \varphi'^2 \right) \\
&= \frac{\pi^2}{2\mu} + \frac{Y}{2} \varphi'^2.
\end{aligned}$$

Now that we have the Hamiltonian, we can work out the Hamilton's equations. The first Hamilton's equation gives,

$$\begin{aligned}
\dot{\varphi}(y) &= \frac{\delta H}{\delta \pi(y)} \\
&= \frac{\delta}{\delta \pi(y)} \int dx \; \left(\frac{\pi^2(x)}{2\mu} + \frac{Y}{2} \varphi'^2(x) \right) \\
&= \int dx \; \frac{\delta}{\delta \pi(y)} \left(\frac{\pi^2(x)}{2\mu} + \frac{Y}{2} \varphi'^2(x) \right) \\
&= \int dx \; \frac{\delta}{\delta \pi(y)} \left(\frac{\pi^2(x)}{2\mu} \right) \\
&= \frac{1}{2\mu} \int dx \; \frac{\delta \pi^2(x)}{\delta \pi(y)} \\
&= \frac{1}{2\mu} \int dx \; 2\pi(x) \; \underbrace{\frac{\delta \pi(x)}{\delta \pi(y)}}_{=\delta(x-y)} \\
&= \frac{1}{\mu} \pi(y).
\end{aligned}$$

The second Hamilton's equation gives,

$$
\begin{aligned}
\dot{\pi}(y) &= -\frac{\delta H}{\delta\varphi(y)} \\
&= -\frac{\delta}{\delta\varphi(y)} \int dx \, \left(\frac{\pi^2(x)}{2\mu} + \frac{Y}{2}\varphi'^2(x)\right) \\
&= -\int dx \, \frac{\delta}{\delta\varphi(y)} \left(\frac{\pi^2(x)}{2\mu} + \frac{Y}{2}\varphi'^2(x)\right) \\
&= -\int dx \, \frac{\delta}{\delta\varphi(y)} \left(\frac{Y}{2}\varphi'^2(x)\right) \\
&= -\frac{Y}{2} \int dx \, \frac{\delta\varphi'^2(x)}{\delta\varphi(y)} \\
&= -\frac{Y}{2} \int dx \, 2\varphi'(x) \, \frac{\delta\varphi'(x)}{\delta\varphi(y)} \\
&= -Y \int dx \, \varphi'(x) \, \left(\frac{\delta\varphi(x)}{\delta\varphi(y)}\right)' \\
&= -Y \int dx \, \varphi'(x) \, \delta'(x-y) \\
&= Y \int dx \, \varphi''(x) \, \delta(x-y) \\
&= Y\varphi''(y).
\end{aligned}
$$

Thus the two Hamilton's equations are

$$\dot{\varphi} = \frac{\pi}{\mu}, \quad \dot{\pi} = Y\varphi''$$

Using the first in the second (eliminating π), we get,

$$\varphi'' - \frac{\mu}{Y}\ddot{\varphi} = 0.$$

Indeed this is the correct equation of motion for the harmonic chain field theory.

3 Poisson Brackets

A Hamiltonian field theory is physical system of infinite degrees of freedom, namely canonically conjugate pairs of fields $\varphi(\boldsymbol{x}), \pi(\boldsymbol{x})$ specified by a Hamiltonian function which is the integral of a density,

$$H = \int d^3\boldsymbol{x} \; \mathcal{H}\left(\pi(\boldsymbol{x}), \varphi(\boldsymbol{x})\right).$$

For a physical system with discrete number of degrees of freedom, say q_i, p_i, $i = 1, \ldots, N$, one can express the time evolution of a physical quantity, $f(q_i, p_i, t)$ as,

$$\frac{df}{dt} = \{f, H\}_{\text{PB}} + \frac{\partial f}{\partial t}$$

where $\{f, H\}_{\text{PB}}$ is the Poisson bracket of f with the Hamiltonian. The Poisson bracket of two quantities, say $A(q_i, p_i)$ and $B(q_i, p_i)$ **is,**

$$\{A, B\}_{\text{PB}} = \sum_i \left(\frac{\partial A}{\partial q_i} \frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i} \frac{\partial B}{\partial q_i} \right). \tag{15}$$

For a field theory, the Poisson bracket (15) can be easily generalized by replacing

$$\sum_i \to \int d^3\boldsymbol{x}$$

$$\frac{\partial}{\partial q_i} \to \frac{\delta}{\delta \varphi(\boldsymbol{x})}$$

$$\frac{\partial}{\partial p_i} \to \frac{\delta}{\delta \pi(\boldsymbol{x})}$$

and we have for a field theory,

$$\{A, B\}_{\text{PB}} = \int d^3\boldsymbol{x} \left(\frac{\delta A}{\delta \varphi(\boldsymbol{x})} \frac{\delta B}{\delta \pi(\boldsymbol{x})} - \frac{\delta A}{\delta \pi(\boldsymbol{x})} \frac{\delta B}{\delta \varphi(\boldsymbol{x})} \right). \tag{16}$$

Thus the time evolution equation for any physical quantity, F which could be either a function or a functional of the canonically conjugate fields, $\varphi(\boldsymbol{x})$, $\pi(\boldsymbol{x})$, is

$$\frac{dF}{dt} = \{F, H\}_{\text{PB}} + \frac{\partial F}{\partial t}, \tag{17}$$

where now,

$$\{F, H\}_{\text{PB}} = \int d^3\boldsymbol{x} \left(\frac{\delta F}{\delta \varphi(\boldsymbol{x})} \frac{\delta H}{\delta \pi(\boldsymbol{x})} - \frac{\delta F}{\delta \pi(\boldsymbol{x})} \frac{\delta H}{\delta \varphi(\boldsymbol{x})} \right).$$

Comment:

- If there is a conserved quantity (charge) in a physical system, i.e if there exists a special functional,

  $$Q = Q\left[\varphi(x), \pi(x), t\right]$$

  such that $\frac{dQ}{dt} = 0$. This implies,

  $$\{Q, H\}_{\text{PB}} = -\frac{\partial Q}{\partial t}.$$

  If there is no explicit time dependence in Q and it is purely a function or functional of $\varphi(\boldsymbol{x}), \pi(\boldsymbol{x})$, then further one has for a conserved charge,

  $$\{Q, H\}_{\text{PB}} = 0. \tag{18}$$

  When we discuss Noether's theorem and conservation laws in field theory, we will make extensive use of this condition (18). Note that this condition is in correspondence with the analogous conservation condition in quantum mechanics,

  $$[Q, H] = 0.$$

Homework

Problem 1. Rederive the equation of motion (7) for a generic theory of a field φ from the functional Euler-:Lagrange equation (1) using the chain rule for functional differentiation.

Problem 2. Derive the Hamilton's equations (14) for a generic theory of a field φ from the functional Hamilton's equations (13) using chain rule for functional differentiation.