PH6418/ EP4618: Quantum Field Theory (Spring 2022)

Notes for Lecture 5-8: Lie algebra of the Poincaré group*

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1 Lorentz group and the Poincaré group

By Lorentz group here we mean only the restricted Lorentz group, $\Lambda_+^\uparrow$, which is the component of the Lorentz group continuously connected to the identity. The restricted Lorentz group of transformations consists of rotations and boosts. The Lorentz group elements are defined by the condition,

$$\Lambda^T \eta \Lambda = \eta, \tag{1}$$

where $\eta = \text{diag}\,(1, -1, -1, -1)$. In $3+1$-dimensions (i.e. 3 space and 1 time dimensions), the Lorentz transformation matrices are 4×4 matrices and hence contain $4 \times 4 = 16$ real elements. Then the defining condition (1), which is a symmetric matrix, leads to $\frac{4(4+1)}{2} = 10$ constraint equations to satisfied by the Lorentz transformation matrix elements. Thus, the total number of independent elements (parameters) in the Lorentz matrix are,

$$16 - 10 = 6.$$

These 6 independent Lorentz matrix parameters are the 3 rotation parameters (angles) and 3 boost parameters (velocities or rapidities).

The Poincaré group consists of the group of Lorentz transformations, *and* the group of translations (shifts of origin of spacetime origin),

$$x^\mu \to x'^\mu = x^\mu + a^\mu. \tag{2}$$

The shift parameters i.e. the a^μ's for $\mu = 0, 1, 2, 3$ can take any real value i.e. they are unconstrained and hence the group of translations is a 4-parameter group and we will denote the translation group as $\mathbb{R}^4$. The translations are an *abelian* group with the group operation being simple addition. The identity element is the zero vector, $(0, 0, 0, 0)$, and the inverse of the element a^μ is the element $-a^\mu$. Thus, the number of parameters of the Poincaré group in $3+1$-dimensions is

$$6 + 4 = 10.$$

The Poincaré group is denoted by the notation $ISO(1, 3)$, and a Poincaré group element is denoted by,

$$P = (\Lambda, a)$$

which acts on points on Minkowski space as,

$$x \to x' = \Lambda\, x + a. \tag{3}$$

The Poincaré group composition law can be deduced by looking at two successive transformations,

$$x \to x' = \Lambda_1 x + a_1,$$

*Typos and errors should be reported to sroy@phy.iith.ac.in and,

$$\begin{aligned} x' \to x'' &= \Lambda_2 x' + a_2 \\ &= \Lambda_2 (\Lambda_1 x + a_1) + a_2 \\ &= \Lambda_2 \Lambda_1 x + \Lambda_2 a_1 + a_2 \\ &= \Lambda x + a, \end{aligned}$$

where, $\Lambda = \Lambda_2 \Lambda_1$ and $a = \Lambda_2 a_1 + a_2$. Thus the group composition rule is,

$$(\Lambda_2, a_2) . (\Lambda_1 a_1) = (\Lambda_2 \Lambda_1,\ \Lambda_2 a_1 + a_2) .$$

Since, the translations group, $\mathbb{R}^4$ is the normal subgroup[1], we conclude that the Poincaré group is a semi-direct product of $\mathbb{R}^4$ and the subgroup of Lorentz transformations:

$$ISO(1,3) = \mathbb{R}^4 \rtimes O(1,3). \tag{4}$$

Evidently the Poincaré transformation (3) is inhomogeneous because the lhs is linear in x' while the rhs is a sum of a term which is linear in x and the zeroth order in x. However one can make the Poincaré transformation appear homogeneous and linear acting on 5-dimensional vectors, $X^A = (x^\mu, 1)$ i.e. the first four components are those of the usual 4-vector, x^μ and the fifth component is just unity, 1.

$$\begin{aligned} X \to X' =& P\, X \\ \begin{pmatrix} x^\mu \\ 1 \end{pmatrix} \to \begin{pmatrix} x'^\mu \\ 1 \end{pmatrix} =& \begin{pmatrix} \Lambda^\mu{}_\nu & a^\mu \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x^\nu \\ 1 \end{pmatrix}. \end{aligned} \tag{5}$$

More generally in $d+1$-dimensions (d space and 1 time), the number of Lorentz boosts is d while the number of rotations is, $\binom{d}{2}$ i.e. total $d + \frac{d(d-1)}{2} = \frac{d(d+1)}{2}$ Lorentz transformations. The number of spacetime translations is $d+1$. Thus the Poincaré group, $ISO(1,d)$ is $d + \frac{d(d+1)}{2}$.

2 Lie Groups and Lie Algebras

Both the Lorentz group and Poincaré groups are *Lie groups*. In this section we familiarize ourselves with some of the key concepts about Lie groups before we delve deeper into the representations of the Lorentz and Poincaré groups.

- ***Continuous groups***: Groups whose elements are labeled by continuous parameter(s) taking values on $\mathbb{R}$ or a subset of $\mathbb{R}$ are called continuous groups i.e. $G = \{g(\theta_1, \theta_2, \ldots)\}$ where θ_i's are continuous parameters. Clearly the total number of elements in a continuous group is infinite. The number of continuous parameters which label each element is called the *dimension* of the group. E.g. in the case of the group of rotations around some axis say, $\hat{\boldsymbol{n}}$ is labeled by the angle of rotation, namely, $R_{\hat{\boldsymbol{n}}}(\theta)$.

- ***Lie Groups***: Continuous groups whose elements are analytic functions of the continuous parameters labeling the elements are called Lie groups. Being analytic functions the group, the group elements can be expressed as convergent Taylor series in the parameters, namely,

$$g(\{\theta_i\}) = g(\{0\}) + \frac{\partial}{\partial \theta_j} g(\{\theta_i\}) \bigg|_{\theta=0} \theta_j + \frac{1}{2!} \frac{\partial^2}{\partial \theta_j \partial \theta_k} g(\{\theta_i\}) \bigg|_{\theta=0} \theta_j \theta_k + \ldots . \tag{6}$$

By convention, we will choose $g(\{0\}) = \mathbb{I}$ i.e. the element at the origin of parameter space to be

[1] It is easy to verify this. The conjugate of a translation element is

$$(\Lambda, a)^{-1} (\mathbb{I}, b) (\Lambda, a) = (\Lambda^{-1}, -a) (\mathbb{I}, b) (\Lambda, a) = (\Lambda^{-1}, -a) (\Lambda, b + a) = (\mathbb{I}, \Lambda^{-1}(b+a) - a) ,$$

which again is a translation subgroup element.

identity, or to be more precise we will label the identity element as the origin of the coordinate system. E.g. consider the group $U(1)$ which is the group of phase transformations consisting of elements, $e^{i\theta}$ where the parameter, $\theta \in [0, 2\pi]$[2] with the group operation being simple multiplication. Clearly such an element can be expanded in a convergent power series,

$$e^{i\theta} = 1 + \theta + \frac{1}{2!}\theta^2 + \ldots .$$

- ***Examples*** of Lie Groups**:** The most common Lie groups which appear in physics literature are $GL(n)$, $SL(n)$, $U(n)$, $SU(n)$, $O(n)$, $SO(n)$, $Sp(n, \mathbb{R})$. We define them here: $GL(n)$ is the group of *bijective linear transformations* acting on a the a given vector space, V. In particular, if $V = \mathbb{R}^n$ i.e. n-dimensional real vector space, we call the particular group of bijective linear transformations to be $GL(n, \mathbb{R})$ and if $V = \mathbb{C}^n$, i.e. n-dimensional complex vector space, we call the special case, $GL(n, \mathbb{C})$. Acting on n-dimensional column vectors, $GL(n, \mathbb{R})$ or $GL(n, \mathbb{C})$ are expressed in the form of $n \times n$ *dimensional invertible matrices* with the group operation being simple matrix multiplication. If in addition these $n \times n$ matrices, have unit determinant then they form a subgroup (i.e. a group of real or complex $n \times n$ dimensional matrices with unit determinant under matrix multiplication), which is called $SL(n, \mathbb{R})$ or $SL(n, \mathbb{C})$.
$$SL(n) \cong GL(n) \cap \det = 1.$$
Thus, $GL(n, \mathbb{R})$ or $GL(n, \mathbb{C})$ or $SL(n, \mathbb{C})$ or $SL(n, \mathbb{R})$ are all matrix Lie groups, the group elements being $n \times n$ matrices whose entries are functions of real parameters[3]. $U(n)$ is the group of *unitary* (norm preserving) bijective linear transformations acting on an n-dimensional complex vector space. Again it can be thought of as a matrix Lie group defined as the group of unitary $n \times n$ matrices,
$$U^\dagger U = \mathbb{I}$$
while $SU(n)$ is a subgroup of $U(n)$ whose elements have unit determinant,
$$SU(n) \cong U(n) \cap \det = 1.$$
Similarly, $O(n)$ is defined to be the group of *orthogonal*(norm preserving) bijective linear transformations acting on an n-dimensional real vector space and can be thought of as a matrix Lie group defined as the group of orthogonal $n \times n$ matrices,
$$O^T O = \mathbb{I},$$
and then $SO(n)$ is defined as the subgroup of $O(n)$ with unit determinant,
$$SO(n) \cong O(n) \cap \det = 1.$$
Finally, $Sp(n, \mathbb{R})$ is defined to be the group of $n \times n$ matrices, say M which preserve the symplectic metric (two-form) Ω i.e.,
$$M^T \, \Omega \, M = \Omega$$
where Ω is a $2n \times 2n$ antisymmetric matrix, defined as,
$$\Omega = \begin{pmatrix} 0 & \mathbb{I}_n \\ -\mathbb{I}_n & 0 \end{pmatrix}.$$

[2] Here 2π is identified with 0.

[3] For $GL(n, \mathbb{C})$ there are n^2 matrix elements, all of which are complex and hence the total number of real parameters needed to constitute an element of $GL(n, \mathbb{C})$ is $2n^2$. Now, an element of $SL(n, \mathbb{C})$ also needs $2n^2$ real parameters to specify it, but the matrix has unit determinant, which imposes a complex equation,
$$|M| = 1 + i.0$$
i.e. two real constraint equations on the total $2n^2$ parameters, thus leaving $\left(2n^2 - 2\right)$ independent real parameters to specify an $SL(n, \mathbb{C})$ matrix. Similar considerations lead to the fact that the dimension of $GL(n, \mathbb{R})$ is n^2 while the dimension of $SL(n, \mathbb{R})$ is $(n^2 - 1)$.

- ***Lie Groups as Riemannian manifolds***: It is natural to ask if the parameter space of a Lie group constitute a manifold, and the answer is yes indeed. The parameter space of Lie groups turn out to be Riemannian manifolds which can be identified with the respective group. The points on the manifold are then identified with the group elements. E.g., the group $U(1)$ is identified with its parameter space, $[0, 2\pi]$ with the end points identified i.e. the unit circle, S^1.

$$U(1) \cong S^1.$$

Similarly one can show that $SU(2) \cong S^3$ while, $SO(3) \cong RP^3$.

- ***Generators*** of a Lie group: One can formally define the define the *generators* of a Lie group by the formula,

$$T_i = i \left.\frac{\partial}{\partial \theta_i} g\left(\{\theta_j\}\right)\right|_{\{\theta_j\}=\{0\}} . \tag{7}$$

The i is a matter of convention, often omitted in math books on Lie groups. It is evident from the definition that the generator is an element of the tangent space of the group manifold at the origin,

$$T_i \in T_O G,$$

where the point O is the origin of the parameter space, $\{\theta_i\} = \{0\}$. If the Lie group, G is n-dimensional, then the tangent space $T_O G$ is spanned by a basis made up of the generators, $\{T_t\}$, $i = 1, \ldots, n$. E.g. Consider the rotation group in 3 dimensions, i.e. $SO(3)$. This can be thought as the set of rotation matrices which leave the length of a 3-vector, (x^1, x^2, x^3) in $\mathbb{R}^3$ unchanged. Consider an element of this group which represents a rotation of the 3-vector around z-axis (3-axis) by an angle θ which is given the following familiar 3×3 matrix

$$R_3\left(\theta\right) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The generator for the rotation around z or 3-axis can then be computed using the definition (7) to be,

$$T_3 = i \left.\frac{\partial}{\partial \theta} R_3(\theta)\right|_{\theta=0} = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{8}$$

Similarly one can work out,

$$T_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, T_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}. \tag{9}$$

- ***The Exponential form (map)*** of Lie group elements: We limit our discussion to matrix Lie groups here but the results applies to generally to all Lie groups. Taking the identity element of a Lie group to be the origin of the parameter space (Riemannian manifold), one can expand an element in a small (infinitesimal) neighborhood of the identity in a convergent Taylor expansion (6) and restrict it to linear order in the parameter:

$$\begin{aligned} g\left(\{\delta\theta_i\}\right) &\approx \underbrace{g\left(\{0\}\right)}_{\mathbb{I}} + \left(\underbrace{\left.\frac{\partial}{\partial \theta_j} g\left(\{\theta_i\}\right)\right|_{\theta_i=0}}_{-i\,T_j} \right) \delta\theta_j \\ &\approx \mathbb{I} - iT_j\,\delta\theta_j. \end{aligned}$$

This equation represents a curve connecting the elements $g(\{\delta\theta_i\})$ and the identity element, $\mathbb{I}$. This curve can be extended to finite (large) values of the parameters i.e. θ_i instead of $\delta\theta_i$ using the exponential map,

$$g\{(\theta_i)\} = \exp(-i\, T_i\, \theta_i) \tag{10}$$

where we have introduced the exponential of a matrix, A to be $\exp A = \sum_{m=1}^{\infty} \frac{A^m}{m!}$. Analyticity guarantees convergence of this infinite series. This exponential map represents a continuous curve on the group manifold with starting point being the origin (Identity) and the end point being the element labeled by the parameters $\{\theta_i\}$.This exponential form of a general group element turns out to be of great utility as will be evident in the following.

- ***Lie algebra of the generators of a Lie group*:** The fact that Lie group elements can be expressed in the exponential form (10), leads to a very important consequence - the tangent space of generators turns into an algebra (called the Lie algebra). Consider the product of two elements, $g_1(\{\theta_i\})$ and $g_2(\{\phi_i\})$ given by two different set of values of the parameters, $\{\theta_i\}$ and $\{\phi_i\}$ where the index i runs over all the parameters, $i = 1, ..., n$. Since by closure law, the product of any two elements of a group elements also belongs to the group, we can write:

$$g_1 g_2 = g_3.$$

Now writing each group element in the exponential form,

$$\exp(-i\, T_i\, \theta_i)\, \exp(-i\, T_j\, \phi_j) = \exp(-i\, T_i\, \psi_i) \tag{11}$$

where we have expressed g_3 also in exponential with the parameters, $\{\psi_i\}$ which are functions of $\{\theta_i, \phi_j\}$. Now the product of two exponential of matrices in the LHS can be turned into a single exponential using the Baker-Campbell-Hausdorff lemma,

$$\exp(A)\, \exp(B) = \exp\left(A + B + \frac{1}{2}[A,B] + \frac{1}{12}[A,[A,B]] - \frac{1}{12}[[A,B],B] + \ldots\right)$$

where the $\ldots$ represent higher order commutators of the matrices A, B. Thus according to the BCH lemma, the equation (11) should look something like,

$$\exp\left(-i\, T_i(\theta_i + \phi_i) + \frac{1}{2}\theta_i\phi_j\,[T_i, T_j] + \frac{1}{12}\theta_i\theta_j\phi_k\,[T_i,[T_j,T_k]] - \frac{1}{12}\theta_i\phi_j\phi_k\,[[T_i,T_j],T_k] + \ldots\right) = \exp(-i\, T_i\, \psi_i)$$

or,

$$-i\, T_i(\theta_i + \phi_i) + \frac{1}{2}\theta_i\phi_j\,[T_i, T_j] - \frac{i}{12}\theta_i\theta_j\phi_k\,[T_i,[T_j,T_k]] + \frac{i}{12}\theta_i\phi_j\phi_k\,[[T_i,T_j],T_k] + \ldots = -i\, T_i\, \psi_i.$$

This matrix equation or (rather vector equation with T_i's lying in the tangent space) can only hold iff if the repeated commutators of generators are all proportional something which is linear in the generators, i.e.,

$$[T_i, T_j] = i\, f_{ij}{}^{k}\, T_k \tag{12}$$

where, the i on the rhs is again a convention followed in physics and $f_{ij}{}^{k}$ are real numbers called the *structure constants* of a Lie group. This special relation among the generators of a Lie group via the commutator is called the Lie algebra of the group. The reason we call it an algebra because the commutator brackets define a product rule in the tangent space, and any vector space with a product rule is called an algebra. In particular, an algebra is called a Lie algebra if the Lie product satisfies three criteria, namely
i. antisymmetry

$$[T_i, T_j] = -[T_j, T_i],$$
$$(\Rightarrow f_{ij}{}^{k} = -f_{ji}{}^{k})$$

ii. Bilinearity

$$[aT_i + bT_j, T_k] = a\,[T_i, T_k] + b\,[T_j, T_k]\,,$$
$$[T_i, aT_j + bT_k] = a\,[T_i, T_j] + b\,[T_i, T_k]\,,$$

and iii. the Jacobi identity

$$[[T_i, T_j]\,, T_k] + [[T_j, T_k]\,, T_i] + [[T_k, T_i]\,, T_j] = 0.$$

For the generators of the group $SO(3)$ worked out before (8, 9), one can check that the Lie algebra is,

$$[T_1, T_2] = iT_3, [T_2, T_3] = iT_1, [T_3, T_1] = iT_2,$$

or,

$$[T_i, T_j] = i\epsilon_{ijk}T_k.$$

So the structure constants for the case of $SO(3)$ are

$$f_{ij}{}^{k} = \epsilon_{ijk},$$

antisymmetric in all three indices.

- ***Representations*** of a Lie group: Lie groups are abstract objects but they can be realized as linear operators (tensors) acting on real or complex vector spaces[4]. Such linear operators (tensors) are called a representation of the Lie group. In physics contexts this vector space could be some vector or matrix denoting the state of the physical system, say position vector, momentum vector, electric or magnetic field vector or some generic tensor, say $F^{ABC\ldots}$, then the group representation is given by the tensor/matrix $D\left(\{\theta_i\}\right)$ and the action on the physical system (vector space of all $F^{ABC\ldots}$'s) is given by the linear transformation rule:

$$F'^{ABC\ldots} = \left[D(\{\theta\})\right]^{ABC\ldots}{}_{PQR\ldots}\, F^{PQR\ldots} \tag{13}$$

and the group multiplication rule is realized as,

$$\left[D\left(\{\theta\}\right)\cdot D\left(\{\phi\}\right)\right]^{ABC\ldots}{}_{PQR\ldots} = \left[D(\{\theta\})\right]^{ABC\ldots}{}_{LMN\ldots}\quad \left[D(\{\phi\})\right]^{LMN\ldots}{}_{PQR\ldots} \tag{14}$$

The labels $ABC\ldots$ could be in general discrete as well as continuous. In the special situation when the physical field has one index (i.e. F^A), the above formulae reduce to,

$$F'^{A} = \left[D(\{\theta\})\right]^{A}{}_{P}\, F^{P},$$
$$\left[D\left(\{\theta\}\right)\cdot D\left(\{\phi\}\right)\right]^{A}{}_{P} = \left[D(\{\theta\})\right]^{A}{}_{L}\; \left[D(\{\phi\})\right]^{L}{}_{P}.$$

Thus in this special case we see that the group element is a two-index object or a matrix, i.e. a ***matrix representation*** of the Lie group under discussion. The form of the generators acting on a generic representation of a Lie group G, say the representation in (13), (14) can be easily extracted by recalling that group elements can be expressed in the exponential form,

$$D(\{\theta\}) = \exp\left(-i\;\theta_j\;T_j\right).$$

Then near $\{\theta\} = 0$, i.e. for infinitesimal parameters, say $\{\delta\theta\}$, one has the expansion to linear order,

$$\begin{aligned}\left[D(\{\delta\theta\})\right]^{ABC\ldots}{}_{PQR\ldots} &= \left(\mathbb{I} - i\;\delta\theta_j\;T_j\right)^{ABC\ldots}{}_{PQR}\\ &= \delta^A_P\;\delta^B_Q\;\delta^C_R - i\;\delta\theta_j\;\left[T_j\right]^{ABC\ldots}{}_{PQR\ldots}.\end{aligned} \tag{15}$$

From this linear order expansion around small parameters one can read off the generators, T_j from the coefficient of the linear term.

[4]In math lingo, a represenation of a (Lie) group is a *homomorphism* from a (Lie) group to the set of bijective linear transformations (operators) of some vector space. Homomorphism is a map which respects the group multiplication rule. Say $D(g_1)$ and $D(g_2)$ are two linear operators which are representations of the elements, g_1 and g_2 of a (Lie) group, then

$$D(g_1)\,D(g_2) = D(g_1 g_2).$$

- An important result from the theory of group representations is that:

  Compact Lie groups have finite dimensional unitary representations

  which implies one can realize compact Lie group elements as finite dimensional matrices which are also unitary! In particular, the exponential form (10) then immediately implies that the generators of the Lie group must be hermitian! This is evident in the example for the compact group $SO(3)$ whose generators (8,9) are all hermitian. Conversely, if a Lie group is non-compact, then it cannot have finite dimensional unitary matrix representations, but instead has non-unitary finite matrix representations, and the generators are given by finite dimensional *anti-hermitian* matrices. We will see an example of this for the non-compact Lorentz group.

3 Lorentz group generators and the Lorentz Algebra

The (restricted) Lorentz group is made up of rotations and boosts. Let's work out the rotation and boost generators using the formula for Lie group generators,

$$T_i = i\,\frac{\partial g\left(\{\theta_j\}\right)}{\partial \theta_i}\bigg|_{\{\theta_j\}=0}, \tag{16}$$

where $g\left(\{\theta_j\}\right)$ is an element of the Lie group for the parameter set $\{\theta_j\}$. First consider the rotations, the Lorentz transformation matrix for rotation by angle θ_3 around the z-axis or 3-axis i.e. in the xy-plane (or 12-plane),

$$R(\theta_3) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_3 & -\sin\theta_3 & 0 \\ 0 & \sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The generator for rotation around 3-axis is,

$$J_3 = i\,\frac{\partial R(\theta_3)}{\partial \theta_3}\bigg|_{\theta_3=0} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{17}$$

Similarly for rotation around the x-axis or 1-axis by an angle θ_1, the group element is

$$R(\theta_1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta_1 & -\sin\theta_1 \\ 0 & 0 & \sin\theta_1 & \cos\theta_1 \end{pmatrix},$$

and the generator,

$$J_1 = i\,\frac{\partial R(\theta_1)}{\partial \theta_1}\bigg|_{\theta_1=0} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}. \tag{18}$$

Finally for the rotation around the y-axis or 2-axis by an angle θ_2 the Lorentz group element is,

$$R(\theta_2) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_2 & 0 & \sin\theta_2 \\ 0 & 0 & 1 & 0 \\ 0 & -\sin\theta_2 & 0 & \cos\theta_2 \end{pmatrix},$$

and its generator is,

$$J_2 = i\left.\frac{\partial R(\theta_2)}{\partial \theta_2}\right|_{\theta_2=0} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}. \tag{19}$$

- Evidently, J_1, J_2, J_3 are all *hermitian* matrices. This is due to the fact that the rotation group is compact, since the range of the group parameters angles $(\theta_1, \theta_2, \theta_3)$ is, $[0, 2\pi]$ is closed.

- One can check by performing matrix multiplications explicitly that these rotation generators, J_1, J_2, J_3 satisfy the $SO(3)$ algebra,

$$[J_1, J_2] = iJ_3,$$
$$[J_2, J_3] = iJ_1,$$
$$[J_3, J_1] = iJ_2.$$

In abstract index notation,

$$[J_i, J_j] = i\epsilon_{ijk}J_k. \tag{20}$$

Next consider the boost transformations, e.g. a boost transformation along the x-direction (1-direction) by a velocity, v is,

$$B_1(\beta) = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

with $\beta = v/c, \gamma = \left(1-\beta^2\right)^{-1/2}$. We will write this in another form, in terms of the rapidity variable, η defined by, $\cosh\eta = \gamma$, $\sinh\eta = \gamma\beta$,

$$B_1(\eta) = \begin{pmatrix} \cosh\eta & \sinh\eta & 0 & 0 \\ \sinh\eta & \cosh\eta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The generator for this is,

$$K_1 = i\left.\frac{\partial B_1(\eta)}{\partial \eta}\right|_{\eta=0} = \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{21}$$

Similarly the generators for the y-boost, K_2 and z-boost, K_3 are,

$$K_2 = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad K_3 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}. \tag{22}$$

- Notice that these matrices are *anti-hermitian*,

$$K^\dagger_{1,2,3} = -K_{1,2,3}.$$

This is due to the fact that the parameter space of boosts is non-compact, $\beta \in [0, 1)$ or $\eta \in [0, \infty)$. The upper end of the interval is not included boosting by speed of light, $v = c$.

- The (Lie) algebra of the boost generators are,

$$[K_1, K_2] = -iJ_3$$
$$[K_2, K_3] = -iJ_1$$
$$[K_3, K_1] = -iJ_2$$

Thus the Lie algebra of the boosts does not close on itself, reminding us that the composition of two boosts in two different directions is not a boost in a third direction but a *mixture of a boost and a rotation!*. In index notation,

$$[K_i, K_j] = -i\epsilon_{ijk} J_k. \tag{23}$$

- Finally the mixed boost-rotation commutators are:

$$[J_1, K_1] = [J_2, K_2] = [J_3, K_3] = 0,$$

$$[J_1, K_2] = iK_3, [J_1, K_3] = -iK_2$$
$$[J_2, K_3] = iK_1, [J_2, K_1] = -iK_3,$$
$$[J_3, K_1] = iK_2, [J_3, K_2] = -iK_1.$$

In abstract index notation,

$$[J_i, K_j] = i\epsilon_{ijk} K_{k.} \tag{24}$$

3.1 Covariant version of the Lorentz generators and the Lorentz algebra

First let's extract the Lorentz generators acting on the Minkowski space ($\mathbb{R}^{1,3}$) points i.e. position 4-vectors, x^μ, instead of those acting on the space of scalar or vector fields,

$$\begin{aligned}
x'^\alpha &= \Lambda^\alpha{}_\beta \, x^\beta \\
&\approx (\delta^\alpha{}_\beta + \omega^\alpha{}_\beta) \, x^\beta \\
&\approx \left(\delta^\alpha{}_\beta + \omega^{\mu\nu} \delta^\alpha_\mu \eta_{\nu\beta}\right) x^\beta \\
&\approx \left(\delta^\alpha{}_\beta + \omega^{\mu\nu} \delta^\alpha_\mu \eta_{\nu\beta}\right) x^\beta \\
&\approx \left[\delta^\alpha{}_\beta + \frac{1}{2}\omega^{\mu\nu} \left(\delta^\alpha_\mu \eta_{\nu\beta} - \delta^\alpha_\nu \eta_{\mu\beta}\right)\right] x^\beta \\
&\approx \left[\delta^\alpha{}_\beta - \frac{i}{2}\omega^{\mu\nu} \left(M_{\mu\nu}\right)^\alpha{}_\beta\right] x^\beta, \quad \left(M_{\mu\nu}\right)^\alpha{}_\beta = i\left(\delta^\alpha_\mu \eta_{\nu\beta} - \delta^\alpha_\nu \eta_{\mu\beta}\right) \\
&\approx \left[\mathbb{I} - \frac{i}{2}\omega^{\mu\nu} M_{\mu\nu}\right]^\alpha{}_\beta \, x^\beta.
\end{aligned}$$

Thus, the generator of Lorentz transformation on the vector space, $\mathbb{R}^{3,1}$ is,

$$\left(M_{\mu\nu}\right)^\alpha{}_\beta = i\left(\delta^\alpha_\mu \eta_{\nu\beta} - \delta^\alpha_\nu \eta_{\mu\beta}\right). \tag{25}$$

Comparing with the forms of the Lorentz transformation matrices, $R(\theta_i)$'s and Lorentz generators defined in the previous section, namely, the J_i's, with the covariant looking formulas (29), (25)we identify,

$$\omega^{12} = -\omega^1{}_2 = \theta_3, \quad \omega^{23} = -\omega^2{}_3 = \theta_1, \quad \omega^{31} = -\omega^3{}_1 = \theta_2,$$

and hence,

$$M_{12} = J_3, M_{23} = J_1, M_{31} = J_2.$$

In abstract index notation,

$$M_{ij} = \epsilon_{ijk} J_k. \tag{26}$$

Similarly, comparing the boost matrices, B_i's and boost generators K_i with the respective covariant versions, (29), (25), we identify,

$$\omega^{01} = -\omega^{0}{}_{1} = \eta$$

and, hence

$$M_{01} = K_1$$

etc.. So we have in general,

$$M_{0i} = K_i. \tag{27}$$

Finally let's work out the covariant version of the Lorentz algebra,

$$\begin{aligned}
[M_{\mu\nu}, M_{\rho\sigma}]^{\alpha}{}_{\beta} &= (M_{\mu\nu})^{\alpha}{}_{\gamma} (M_{\rho\sigma})^{\gamma}{}_{\beta} - (\mu \leftrightarrow \rho, \nu \leftrightarrow \sigma) \\
&= i\left(\delta^{\alpha}_{\mu}\eta_{\nu\gamma} - \delta^{\alpha}_{\nu}\eta_{\mu\gamma}\right) i\left(\delta^{\gamma}_{\rho}\eta_{\sigma\beta} - \delta^{\gamma}_{\sigma}\eta_{\rho\beta}\right) - (\mu \leftrightarrow \rho, \nu \leftrightarrow \sigma) \\
&= i^2\left(\delta^{\alpha}_{\mu}\eta_{\nu\gamma}\delta^{\gamma}_{\rho}\eta_{\sigma\beta} - \delta^{\alpha}_{\nu}\eta_{\mu\gamma}\delta^{\gamma}_{\rho}\eta_{\sigma\beta} - \delta^{\alpha}_{\mu}\eta_{\nu\gamma}\delta^{\gamma}_{\sigma}\eta_{\rho\beta} + \delta^{\alpha}_{\nu}\eta_{\mu\gamma}\delta^{\gamma}_{\sigma}\eta_{\rho\beta}\right) - (\mu \leftrightarrow \rho, \nu \leftrightarrow \sigma) \\
&= i^2\left(\delta^{\alpha}_{\mu}\eta_{\nu\rho}\eta_{\sigma\beta} - \delta^{\alpha}_{\nu}\eta_{\mu\rho}\eta_{\sigma\beta} - \delta^{\alpha}_{\mu}\eta_{\nu\sigma}\eta_{\rho\beta} + \delta^{\alpha}_{\nu}\eta_{\mu\sigma}\eta_{\rho\beta}\right) - (\mu \leftrightarrow \rho, \nu \leftrightarrow \sigma) \\
&= -i\eta_{\mu\rho} i\left(\delta^{\alpha}_{\nu}\eta_{\sigma\beta}\right) - i\eta_{\nu\sigma} i\left(\delta^{\alpha}_{\mu}\eta_{\rho\beta}\right) + i\eta_{\nu\rho} i\left(\delta^{\alpha}_{\mu}\eta_{\sigma\beta}\right) + i\eta_{\mu\sigma} i\left(\delta^{\alpha}_{\nu}\eta_{\rho\beta}\right) - (\mu \leftrightarrow \rho, \nu \leftrightarrow \sigma) \\
&= -i\eta_{\mu\rho} \underbrace{i\left(\delta^{\alpha}_{\nu}\eta_{\sigma\beta} - \delta^{\alpha}_{\sigma}\eta_{\nu\beta}\right)}_{=(M_{\nu\sigma})^{\alpha}{}_{\beta}} - i\eta_{\nu\sigma} \underbrace{i\left(\delta^{\alpha}_{\mu}\eta_{\rho\beta} - \delta^{\alpha}_{\rho}\eta_{\mu\beta}\right)}_{=(M_{\mu\rho})^{\alpha}{}_{\beta}} + i\eta_{\nu\rho} \underbrace{i\left(\delta^{\alpha}_{\mu}\eta_{\sigma\beta} - \delta^{\alpha}_{\sigma}\eta_{\mu\beta}\right)}_{=(M_{\mu\sigma})^{\alpha}{}_{\beta}} + i\eta_{\mu\sigma} \underbrace{i\left(\delta^{\alpha}_{\nu}\eta_{\rho\beta} - \delta^{\alpha}_{\rho}\eta_{\nu\beta}\right)}_{=(M_{\nu\rho})^{\alpha}{}_{\beta}} \\
&= \left(-i\eta_{\mu\rho}M_{\nu\sigma} - i\eta_{\nu\sigma}M_{\mu\rho} + i\eta_{\nu\rho}M_{\mu\sigma} + i\eta_{\mu\sigma}M_{\nu\rho}\right)^{\alpha}{}_{\beta}.
\end{aligned}$$

Thus we have the covariant version of the Lorentz algebra,

$$[M_{\mu\nu}, M_{\rho\sigma}] = -i\eta_{\mu\rho}M_{\nu\sigma} - i\eta_{\nu\sigma}M_{\mu\rho} + i\eta_{\nu\rho}M_{\mu\sigma} + i\eta_{\mu\sigma}M_{\nu\rho}. \tag{28}$$

One can check that this covariant form indeed reproduces the Lie algebra of the Lorentz group derived in the last section. E.g.

$$\begin{aligned}
&[M_{01}, M_{02}] = -i\eta_{00}M_{12} - i\eta_{12}M_{00} + i\eta_{10}M_{02} + i\eta_{02}M_{10} \\
&\Rightarrow [K_1, K_2] = -iJ_3,
\end{aligned}$$

or,

$$\begin{aligned}
&[M_{12}, M_{23}] = -i\eta_{12}M_{23} - i\eta_{23}M_{12} + i\eta_{13}M_{22} + i\eta_{22}M_{13} \\
&\Rightarrow [J_3, J_1] = iJ_2.
\end{aligned}$$

or,

$$\begin{aligned}
&[M_{23}, M_{02}] = -i\eta_{20}M_{32} - i\eta_{32}M_{20} + i\eta_{22}M_{30} + i\eta_{30}M_{22} \\
&\Rightarrow [J_1, K_2] = iK_3.
\end{aligned}$$

3.2 Action of the Lorentz group on physical fields and the Lorentz generators in the field space

In this section we are interested in more general representations of the Lorentz group other than the spacetime points, $\mathbb{R}^{1,3}$, namely the scalar and vector fields. Under a proper orthochronous Lorentz transformation,

$$x \to x' = \Lambda . x$$

a physical field, call it $F^{ABC..}(x)$, furnishing a generic representation of the Lorentz group transforms like,

$$F'^{ABC...}(x') = [D(\Lambda)]^{ABC...}{}_{PQR...}F^{PQR...}.$$

For a scalar field which has no indices, $F^{ABC...} = \varphi(x)$, while for the vector field, we have a single Lorentz index, i.e. $A = \mu$, and $F^{ABC...} = V^{\mu}$. To extract the form of the generators of the Lorentz group acting on the scalar and vector space or for that matter a general tensor field, say $F^{ABC\cdot\cdot}(x)$ with a bunch of indices $A, B, C, \ldots$, we will need to use the infinitesimal forms of the transformations namely,

$$\Lambda^{\mu}{}_{\nu} = \delta^{\mu}_{\nu} + \omega^{\mu}{}_{\nu}, \tag{29}$$

and

$$F'^{ABC\ldots}(x) \approx \left(\mathbb{I} - i\,\omega^{\mu\nu}\, M_{\mu\nu}\right)^{ABC\ldots}{}_{PQR\ldots} \quad F^{PQR\ldots}(x). \tag{30}$$

To illustrate the procedure, let's extract the form of the Lorentz generators acting the scalar space and vector space which transform like,

$$\varphi(x) \to \varphi'(x') = \varphi(x), \tag{31}$$

and,

$$V^{\mu}(x) \to V'^{\mu}(x') = \Lambda^{\mu}{}_{\nu}\, V^{\nu}(x) \tag{32}$$

respectively. For the scalar field, which has no indices, we have,

$$\varphi'(x') = \varphi(x), x' = \Lambda\, x$$

or,

$$\begin{aligned}
\varphi'(x) &= \varphi(\Lambda^{-1}x) \\
&= \varphi\left(\left(\Lambda^{-1}\right)^{\mu}{}_{\nu}\, x^{\nu}\right) \\
&\approx \varphi\left((\delta^{\mu}_{\nu} - \omega^{\mu}{}_{\nu})x^{\nu}\right) \\
&\approx \varphi\left(x^{\mu} - \omega^{\mu}{}_{\nu}x^{\nu}\right) \\
&\approx \varphi(x^{\mu}) - \omega^{\mu}{}_{\nu}\, x^{\nu}\partial_{\mu}\varphi(x) \\
&\approx \left[1 - \omega^{\mu\nu}x_{\nu}\partial_{\mu}\right]\varphi(x) \\
&\approx \left[1 - \frac{1}{2}\omega^{\mu\nu}\left(x_{\nu}\partial_{\mu} - x_{\mu}\partial_{\nu}\right)\right]\varphi(x) \\
&\approx \left[1 - \frac{i}{2}\omega^{\mu\nu}\, \underbrace{i\left(x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu}\right)}_{M_{\mu\nu}}\right]\varphi(x).
\end{aligned}$$

Thus we identify the Lorentz generator acting on the scalar function space as the linear operator,

$$M_{\mu\nu} = i\left(x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu}\right). \tag{33}$$

Next for the vector field,

$$V'^{\alpha}(x') = \Lambda^{\alpha}{}_{\beta}\, V^{\beta}(x),$$

or,

$$\begin{aligned}
V'^{\alpha}(x) &= \Lambda^{\alpha}{}_{\beta}\, V^{\beta}\left(\Lambda^{-1}x\right) \\
&= \Lambda^{\alpha}{}_{\beta}\, V^{\beta}\left(\left(\Lambda^{-1}\right)^{\mu}{}_{\nu}\, x^{\nu}\right) \\
&\approx \left(\delta^{\alpha}_{\beta} + \omega^{\alpha}{}_{\beta}\right) V^{\beta}\left((\delta^{\mu}_{\nu} - \omega^{\mu}{}_{\nu})x^{\nu}\right) \\
&\approx \left(\delta^{\alpha}_{\beta} + \omega^{\alpha}{}_{\beta}\right) V^{\beta}\left(x^{\mu} - \omega^{\mu}{}_{\nu}x^{\nu}\right) \\
&\approx \left(\delta^{\alpha}_{\beta} + \omega^{\alpha}{}_{\beta}\right)\left(V^{\beta}(x^{\mu}) - \omega^{\mu}{}_{\nu}x^{\nu}\partial_{\mu}V^{\beta}(x)\right) \\
&\approx V^{\alpha}(x) + \omega^{\alpha}{}_{\beta}\, V^{\beta}(x) - \omega^{\mu}{}_{\nu}x^{\nu}\partial_{\mu}V^{\alpha}(x) \\
&\approx \left(\delta^{\alpha}_{\beta} + \omega^{\alpha}{}_{\beta} - \omega^{\mu\nu}\,\delta^{\alpha}_{\beta}\, x_{\nu}\partial_{\mu}\right) V^{\beta}(x) \\
&\approx \left(\delta^{\alpha}_{\beta} + \omega^{\mu\nu}\delta^{\alpha}_{\mu}\eta_{\nu\beta} - \omega^{\mu\nu}\,\delta^{\alpha}_{\beta}\, x_{\nu}\partial_{\mu}\right) V^{\beta}(x) \\
&\approx \left[\delta^{\alpha}_{\beta} + \frac{1}{2}\omega^{\mu\nu}\left(\delta^{\alpha}_{\mu}\eta_{\nu\beta} - \delta^{\alpha}_{\nu}\eta_{\mu\beta}\right) + \frac{1}{2}\omega^{\mu\nu}\,\delta^{\alpha}_{\beta}\,\left(x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu}\right)\right] V^{\beta}(x) \\
&\approx \left[\mathbb{I} - \frac{i}{2}\omega^{\mu\nu}M_{\mu\nu}\right]^{\alpha}{}_{\beta}\, V^{\beta}(x),
\end{aligned}$$

where, the Lorentz generator has been identified to be,

$$(M_{\mu\nu})^{\alpha}{}_{\beta} = i\left(\delta^{\alpha}_{\mu}\eta_{\nu\beta} - \delta^{\alpha}_{\nu}\eta_{\mu\beta}\right) + i\left(x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu}\right)\delta^{\alpha}_{\beta}. \tag{34}$$

The two extra indices α, β in this expression compared to the expression of the Lorentz generator acting on the scalar fields, (33) indicates that the Lorentz group is acting on a vector field i.e. a field with a Lorentz index. Evidently the part,

$$(L_{\mu\nu})^{\alpha}{}_{\beta} = i\left(x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu}\right)\delta^{\alpha}_{\beta}$$

is the orbital angular momentum part (it depends on the position, x_μ and momentum $\hat{p}_\mu = i\partial_\mu$), while the part,

$$(S_{\mu\nu})^{\alpha}{}_{\beta} = i\left(\delta^{\alpha}_{\mu}\eta_{\nu\beta} - \delta^{\alpha}_{\nu}\eta_{\mu\beta}\right),$$

is the spin angular momentum part (independent of position or momentum).

HW Problem: **Derive/Reproduce the Lie algebra of the Lorentz group (28) for the Lorentz generators, (33) and (34) acting on the scalar and vector spaces respectively.**

4 Including the translation generators: Poincaré algebra

The Lorentz group needs to be augmented by the abelian group of spacetime translations to constitute the Poincaré group. Let's first extract the form of the generators of the Poincaré group acting on spacetime points. To this end, we introduce the 5-dimensional vector, $X^A = (x^\mu, 1)\,, A = 0, 1, \ldots, 4$. The Poincaré transformation on spacetime points,

$$x \to x' = \Lambda\, x - a.$$

is represented by the vector-matrix equation,

$$X \to X' = P\, X,$$

where,

$$P = \begin{pmatrix} (\Lambda^{\mu}{}_{\nu})_{4\times 4} & (-a^{\mu})_{4\times 1} \\ (0)_{1\times 4} & 1 \end{pmatrix},$$

i.e.

$$\begin{aligned} P^{A}{}_{B} &= \delta^{A}_{\mu}\,\delta^{\nu}_{B}\,\underbrace{P^{\mu}{}_{\nu}}_{=\Lambda^{\mu}{}_{\nu}} + \delta^{A}_{\mu}\,\delta^{4}_{B}\,\underbrace{P^{\mu}{}_{4}}_{=-a^{\mu}} + \delta^{A}_{4}\,\delta^{\nu}_{B}\,\underbrace{P^{4}{}_{\nu}}_{=0} + \delta^{A}_{4}\,\delta^{4}_{B}\,\underbrace{P^{4}{}_{4}}_{=1} \\ &= \delta^{A}_{\mu}\,\delta^{\nu}_{B}\,\Lambda^{\mu}{}_{\nu} - \delta^{A}_{\mu}\,\delta^{4}_{B}\,a^{\mu} + \delta^{A}_{4}\,\delta^{4}_{B}. \end{aligned}$$

Now consider infinitesimal form of the transformation, i.e. $\Lambda^{\mu}{}_{\nu} = \delta^{\mu}_{\nu} + \omega^{\mu}{}_{\nu}$ with, $\omega^{\mu}{}_{\nu} \to 0$, and $a^\mu \to 0$. In that case, to first order in the parameters a, ω,

$$\begin{aligned} P^{A}{}_{B} &\approx \delta^{A}_{\mu}\,\delta^{\nu}_{B}\,\left(\delta^{\mu}_{\nu} + \omega^{\mu}{}_{\nu}\right) - \delta^{A}_{\mu}\,\delta^{4}_{B}\,a^{\mu} + \delta^{A}_{4}\,\delta^{4}_{B} \\ &\approx \underline{\delta^{A}_{\mu}\,\delta^{\mu}_{B}} + \delta^{A}_{\mu}\,\delta^{\nu}_{B}\,\omega^{\mu}{}_{\nu} - \delta^{A}_{\mu}\,\delta^{4}_{B}\,a^{\mu} + \underline{\delta^{A}_{4}\,\delta^{4}_{B}} \\ &\approx \delta^{A}_{C}\,\delta^{C}_{B} + \delta^{A}_{\mu}\,\delta^{\nu}_{B}\,\omega^{\mu}{}_{\nu} - \delta^{A}_{\mu}\,\delta^{4}_{B}\,a^{\mu} \\ &\approx \delta^{A}_{C}\,\delta^{C}_{B} + \delta^{A}_{\mu}\,\eta_{\nu B}\,\omega^{\mu\nu} - \delta^{A}_{\mu}\,\delta^{4}_{B}\,a^{\mu} \\ &\approx \delta^{A}_{C}\,\delta^{C}_{B} - \frac{i}{2}\omega^{\mu\nu}\,\underbrace{i\left(\delta^{A}_{\mu}\,\eta_{\nu B} - \delta^{A}_{\nu}\,\eta_{\mu B}\right)}_{(M_{\mu\nu})^{A}{}_{B}} - ia^{\mu}\,\underbrace{\left(-i\delta^{A}_{\mu}\,\delta^{4}_{B}\right)}_{(P_{\mu})^{A}{}_{B}}. \end{aligned}$$

So we get the Lorentz generators,

$$(M_{\mu\nu})^{A}{}_{B} = i\left(\delta^{A}_{\mu}\,\eta_{\nu B} - \delta^{A}_{\nu}\,\eta_{\mu B}\right),$$

and the translation generators,

$$(P_{\mu})^{A}{}_{B} = -i\,\delta^{A}_{\mu}\,\delta^{4}_{B}.$$

Note that here we have introduced a new object, $\eta_{\nu B} = \delta^{\nu}_{B}$. This is only true if B is a greek (Lorentz) index. If $B = 5$, then $\eta_{\nu B} = \delta^{\nu}_{B} = 0$.

The Lie algebra of the Poincaré generators is now easy to work out. We already know what the algebra of the Lorentz generators is. So we only need to work out the translation algebra elements, i.e. $[P_\mu, P_\nu]$ and the mixed algebra elements, $[M_{\mu\nu}, P_\rho]$.

$$\begin{aligned}
[P_\mu, P_\nu]^A{}_B &= (P_\mu)^A{}_C\ (P_\nu)^C{}_B - (\mu \leftrightarrow \nu) \\
&= \left(-i\delta^A_\mu\ \delta^4_C\right)\left(-i\delta^C_\nu\ \delta^4_B\right) - (\mu \leftrightarrow \nu) \\
&= -\delta^A_\mu\ \delta^4_C\ \delta^C_\nu\ \delta^4_B - (\mu \leftrightarrow \nu) \\
&= -\delta^A_\mu\ \underbrace{\delta^4_\nu}_{=0}\ \delta^4_B - (\mu \leftrightarrow \nu) \\
&= 0.
\end{aligned}$$

$$\begin{aligned}
[M_{\mu\nu}, P_\rho]^A{}_B &= (M_{\mu\nu})^A{}_C\ (P_\rho)^C{}_B - (P_\rho)^A{}_C\ (M_{\mu\nu})^C{}_B \\
&= i\left(\delta^A_\mu\ \eta_{\nu C} - \delta^A_\nu\ \eta_{\mu C}\right)\left(-i\delta^C_\rho\ \delta^4_B\right) - \left(-i\delta^A_\rho\ \delta^4_C\right) i\left(\delta^C_\mu\ \eta_{\nu B} - \delta^C_\nu\ \eta_{\mu B}\right) \\
&= -i^2\left(\delta^A_\mu\ \eta_{\nu C}\ \delta^C_\rho\ \delta^4_B - \delta^A_\nu\ \eta_{\mu C}\ \delta^C_\rho\ \delta^4_B\right) + i^2\left(\delta^A_\rho\ \delta^4_C\ \delta^C_\mu\ \eta_{\nu B} - \delta^{\mathbf{A}}_\rho\ \delta^{\mathbf{4}}_{\mathbf{C}}\ \delta^C_\nu\ \eta_{\mu B}\right) \\
&= -i^2\left(\delta^A_\mu\ \eta_{\nu\rho}\ \delta^4_B - \delta^A_\nu\ \eta_{\mu\rho}\ \delta^4_B\right) + i^2\left(\delta^A_\rho\ \underbrace{\delta^4_\mu}_{=0}\ \eta_{\nu B} - \delta^{\mathbf{A}}_\rho\ \underbrace{\delta^4_\nu}_{=\mathbf{0}}\ \eta_{\mu B}\right) \\
&= i\eta_{\nu\rho}\left(-\underbrace{i\ \delta^A_\mu\ \delta^4_B}_{(P_\mu)^A{}_B}\right) - i\eta_{\mu\rho}\left(-\underbrace{i\ \delta^A_\nu\ \delta^4_B}_{=(P_\nu)^A{}_B}\right) \\
&= i\eta_{\nu\rho}\,(P_\mu)^A{}_B - i\eta_{\mu\rho}\,(P_\nu)^A{}_B.
\end{aligned}$$

Thus, we have,

$$\begin{aligned}
[P_\mu, P_\nu] &= 0, \\
[M_{\mu\nu}, P_\rho] &= i\eta_{\rho\nu} P_\mu - i\eta_{\rho\mu} P_\nu.
\end{aligned}$$

Next we extract the form of the translation generator acting on the scalar field (function space) and vector field (function space). For the scalar field, under $x \to x' = x - a$,

$$\begin{aligned}
\varphi'(x') &= \varphi(x) \\
\Rightarrow \varphi'(x - a) &= \varphi(x) \\
\Rightarrow \varphi'(x) &= \varphi(x + a) \\
&\approx \varphi(x) + a^\mu \partial_\mu \varphi(x) \\
&\approx (1 + a^\mu \partial_\mu)\,\varphi(x) \\
&\approx \left[1 - i\,a^\mu\left(\underbrace{i\partial_\mu}_{P_\mu}\right)\right]\varphi(x), \\
\Rightarrow P_\mu &= i\partial_\mu.
\end{aligned}$$

Similarly for a vector field under translation, $x \to x' = x - a$,

$$\begin{aligned}
V'^{\mu}(x') &= V^{\mu}(x), \\
\Rightarrow V'^{\mu}(x-a) &= V^{\mu}(x), \\
\Rightarrow V'^{\mu}(x) &= V^{\mu}(x+a) \\
&\approx V^{\mu}(x) + a^{\alpha}\partial_{\alpha}V^{\mu}(x) \\
&\approx \left(\delta^{\mu}_{\nu} + a^{\alpha}\,\delta^{\mu}_{\nu}\partial_{\alpha}\right)V^{\nu}(x) \\
&\approx \left[\delta^{\mu}_{\nu} - ia^{\alpha}\,\underbrace{\left(i\delta^{\mu}_{\nu}\partial_{\alpha}\right)}_{=[P_{\alpha}]^{\mu}{}_{\nu}}\right]V^{\nu}(x) \\
&\approx \left(\mathbb{I} - i\,a^{\alpha}\,P_{\alpha}\right)^{\mu}{}_{\nu}\,V^{\nu}(x), \\
\Rightarrow [P_{\alpha}]^{\mu}{}_{\nu} &= i\delta^{\mu}_{\nu}\partial_{\alpha}.
\end{aligned}$$

For both these cases the algebra of the translation generators is,

$$[P_{\mu}, P_{\nu}]\,\varphi(x) = i^2\,[\partial_{\mu}, \partial_{\nu}]\,\varphi = 0,$$

$$\begin{aligned}
[P_{\mu}, P_{\nu}]^{\alpha}{}_{\beta}V^{\beta}(x) &= (P_{\mu})^{\alpha}{}_{\gamma}\,(P_{\nu})^{\gamma}{}_{\beta}V^{\beta}(x) - (\mu \leftrightarrow \nu) \\
&= \left(i\delta^{\alpha}_{\gamma}\partial_{\mu}\right)\left(i\delta^{\gamma}_{\beta}\partial_{\nu}\right)V^{\beta}(x) - (\mu \leftrightarrow \nu) \\
&= i^2\,[\partial_{\mu}, \partial_{\nu}]\,V^{\alpha}(x) \\
&= 0.
\end{aligned}$$

Homework: Derive the Lie algebra of the translation generators, P_{μ} and the Lorentz generators i.e. $[M_{\mu\nu}, P_{\rho}]$ on the represenation space of scalar and vector field.