Quantum Field Theory (PH6418/ EP4618): Spring 2022

Notes for lecture 9*

March 28, 2022

1 Lorentz covariant Euler-Lagrange equations for a field theory

We revisit the Euler-Lagrange (EL) equations for field theory of a general (Lorentz tensor or spinor) field, $\mathcal{F}(x)$

$$\frac{\partial \mathcal{L}}{\partial \mathcal{F}} = \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\mathcal{F}}}\right) + \boldsymbol{\nabla}\cdot\left(\frac{\partial \mathcal{L}}{\partial (\boldsymbol{\nabla}\mathcal{F})}\right), \tag{1}$$

which we wrote down for a non-relativistic field theory, defined by the action,

$$I\left[\varphi(x)\right] = \int d^4x \; \mathcal{L}\left(\mathcal{F}(x), \dot{\mathcal{F}}(x), \boldsymbol{\nabla}\mathcal{F}(x)\right).$$

The Lorentz or Dirac indices of $\mathcal{F}$ are not displayed but understood to be present here. Now we write down the Lorentz covariant version of the Euler-Lagrange equation (1). Here the term covariant equation means both sides of the equation are transforming identically under Lorentz (Poincaŕ) transformations. To this end we note that the partial derivatives $\frac{1}{c}\frac{\partial}{\partial t}, \boldsymbol{\nabla}$ can be combined into a covariant vector derivative, $\partial_\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, \boldsymbol{\nabla}\right)$, as well as the derivatives of the Lagrangian-density (scalar) with respect to the field derivatives, $\frac{\partial \mathcal{L}}{\partial \dot{\mathcal{F}}}, \frac{\partial \mathcal{L}}{\partial(\boldsymbol{\nabla}\mathcal{F})}$ can be packaged into

$$V^\mu = \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \mathcal{F}\right)}.$$

It is then evident that the EL equation (1) can be expressed as Lorentz covariant equation,

$$\frac{\partial \mathcal{L}}{\partial \varphi} = \partial_\mu V^\mu,$$

or,

$$\frac{\partial \mathcal{L}}{\partial \mathcal{F}} = \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \mathcal{F}\right)}\right). \tag{2}$$

This is covariant because both sides have free Lorentz indices (or free Dirac indices) coming from the field $\mathcal{F}$ only (the μ index on the RHS here is contracted so is not free) and hence both sides will transform indentically under Lorentz transformations. From now on we will write the Lorentz covariant version of the EL equation, namely,

*Typos and errors should be emailed to the Instructor: Shubho Roy (email: sroy@phy.iith.ac.in)

Comments:

For example for the Maxwell field, A_ν, the EL equation is obtained by plugging $\mathcal{F} \to A_\nu$ in the EL equation (2),

$$\frac{\partial \mathcal{L}}{\partial A_\nu} = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \left(\partial_\mu A_\nu\right)} \right).$$

2 Natural Units and Mass dimensions of physical quantities

From now one we will work in units where

$$c = \hbar = 1. \tag{3}$$

Apart from getting rid of all the annoying factors of c in all formulas, setting $c = 1$ has great consequences as far as dimensional analysis is concerned. Since c is the speed of light it has dimensions same as the ratio of length and time,

$$[c] = \left[\frac{L}{T}\right] = [L] - [T]$$

The consequences of $c = 1$ is that length and time have same dimensions,

$$\begin{aligned} c &= 1 \\ \Rightarrow [c] &= 0, \\ \Rightarrow [L] &= [T]\,. \end{aligned} \tag{4}$$

Father consider the Einstein mass-energy conversion equation, $E = m\,c^2$ and compute dimensions of both sides,

$$\begin{aligned} [E] &= \left[m\,c^2\right] \\ &= [M] + 2\,[c] \\ &= [M]\,. \end{aligned}$$

Similarly, using the relativistic energy-momentum equation for massless particles, $E = p\,c$, we get,

$$[E] = [P]$$

Thus we note that energy, linear momentum and mass all have the same dimensions,

$$[E] = [p] = [M]\,. \tag{5}$$

Now let's look at the consequences for setting $\hbar = 1$ i.e. $[\hbar] = 0$. The Einstein-Planck relation, $E = \hbar\omega$ then gives,

$$\begin{aligned} [E] &= [\hbar\,\omega] \\ &= \underbrace{[\hbar]}_{=0} + \Big[\underbrace{\omega}_{=\frac{1}{T}}\Big] \\ &= \left[\frac{1}{T}\right] \\ &= -\,[T]\,, \end{aligned}$$

or,

$$[T] = -[E]\,. \tag{6}$$

Similarly, from the canonical commutation relation, $[x,p] = i\hbar$, we get,

$$[L] = -[P] \tag{7}$$

Now we adopt the convention, $[M] = 1$**.** Under this convention we have from the above discussion,

$$\begin{aligned}[E] &= [P] = 1, \\ [L] &= [T] = -1.\end{aligned}$$

These dimensions of various physical quantities in natural units when $[M] = 1$, are called the ***mass dimensions* or *energy dimensions***. Let's work out the mass dimensions of some familiar physical quantities. First example is that of the Newtonian concept of force, F. As per Newton's second law, $F = m\,\frac{d^2x}{dt^2}$,

$$\begin{aligned}[F] &= \left[m\,\frac{d^2x}{dt^2}\right] \\ &= [m] + [x] - \left[dt^2\right] \\ &= [M] + [L] - 2\,[T] \\ &= 1 - 1 - 2\,(-1) \\ &= 2.\end{aligned}$$

So the mass dimensions of force is $+2$. Next we ask what is the mass dimension of action, I. The action is defined as the time integral of the Lagrangian,

$$I = \int dt \text{ Lagrangian.}$$

So,

$$\begin{aligned}[I] &= [dt] + [\text{Lagrangian}] \\ &= -1 + 1 \\ &= 0.\end{aligned}$$

So the action is dimensionless. Here we have used the fact the Lagrangian has same dimensions as energy since in classical mechanics is given by the difference in kinetic and potential energy,

$$\text{Lagrangian} = T - V \Rightarrow [\text{Lagrangian}] = [E] = +1.$$

Next, consider angular momentum defined by, $L_i = \epsilon_{ijk} x_j p_k$,

$$\begin{aligned}[L_i] &= [x_j] + [p_k] \\ &= -1 + 1 \\ &= 0!\end{aligned}$$

So the angular momentum is dimensionless as well. The fact that angular momentum and action are dimensionless can also be arrived at by recalling that both these quantities have same dimensions of $\hbar$, and hence in natural units $[\hbar] = 0$.

3 An action functional and the equation of motion for scalar field

Here we consider the field theory where the degree of freedom is a real scalar field, $\varphi(x)$. The first step is to construct an action functional, $I\left[\varphi(x)\right]$. The action must be real, as well as a Lorentz scalar, and since this is a field theory the action must be a spacetime integral of a lagrangian density,

$$I\left[\varphi(x)\right] = \int d^4x\, \mathcal{L}(x),$$

where the lagrangian density is itself a Lorentz scalar and a **local** function of the field φ and the first derivative(s) $\partial_\mu\varphi$,

$$\mathcal{L}(x) = \mathcal{L}\left(\varphi(x), \partial_\mu\varphi(x)\right).$$

Since the lagrangian density is a Lorentz scalar, one can allow terms in the lagrangian density an arbitrary function of the scalar field $\varphi(x)$ itself, say $V(\varphi)$. Now terms in the lagrangian which will contain derivatives of the field, such as $\partial_\mu\varphi$, one needs to construct scalar by contracting the Lorentz indices. One such term is $\left(\partial_\mu\varphi\right)\left(\partial^\mu\varphi\right)$. However generically one can have more general terms such as $K(\varphi)\left(\partial_\mu\varphi\right)\left(\partial^\mu\varphi\right)$, where $K(\varphi)$ is an arbitrary function of the scalar field. So it seems the general lagrangian is of the form,

$$\mathcal{L} = K(\varphi)\left(\partial_\mu\varphi\right)\left(\partial^\mu\varphi\right) - V(\varphi).$$

The first term containing derivatives will be referred to as the *kinetic energy* term,

$$T = K(\varphi)\left(\partial_\mu\varphi\right)\left(\partial^\mu\varphi\right).$$

For now on we will always consider a *canonically normalized* which has a factor of $\frac{1}{2}$, and no dependence on the field φ, i.e. $K(\varphi) = \frac{1}{2}$

$$T = \frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi.$$

With this canonically normalized term, the lagrangian for a real scalar field theory is,

$$\mathcal{L} = T - V = \frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi - V(\varphi).$$

Next we write down the Euler-Lagrange equation for this real scalar field,

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\varphi\right)}\right) = \frac{\partial\mathcal{L}}{\partial\varphi}$$

Inserting, $\frac{\partial\mathcal{L}}{\partial\varphi} = -\frac{\partial V}{\partial\varphi}$ and $\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)} = \partial^\mu\varphi$ respectively, one gets the equation of motion,

$$\Box\varphi = -\frac{\partial V}{\partial\varphi},$$

or,

$$\Box\varphi + \frac{\partial V}{\partial\varphi} = 0.$$

where we are denoting $\partial_\mu \partial^\mu \equiv \Box$.

Now to consider physically interesting theories we need to specialize or restrict ourselves to some specific forms of the potential function, $V(\varphi)$. One physical such restriction is that the potential should be an analytic function in φ. At the level of classical theory one might not be able to appreciate the significance of this restriction, but in quantum theory it will become evident that non-analytic potentials or terms in the action will imply or result in breakdown of effective field theory description at some regions of moduli space. Foreseeing this situation we will strictly consider analytic potentials, $V(\varphi)$ and since analytic functions as a power series,

$$V(\varphi) = \sum_{n=0}^{\infty} a_n\, \varphi^n = a_0 + a_1 \varphi + a_2\, \varphi^2 + a_3\, \varphi^3 + a_4\, \varphi^4 + \dots.$$

We can safely drop or ignore a constant piece a_0 since such a constant term in the lagrangian does not contribute to the equation of motion,

$$V(\varphi) = a_1 \varphi + a_2\, \varphi^2 + a_3\, \varphi^3 + a_4\, \varphi^4 + \dots.$$

For simplicity we will restrict ourselves to quartic terms,

$$V(\varphi) = a_1 \varphi + a_2\, \varphi^2 + a_3\, \varphi^3 + a_4\, \varphi^4.$$

(The higherst power is even in φ and hence the potential is bounded from below which means the theory has a stable ground state). Now one can always redefine the field,

$$\varphi \to \varphi - A$$

for some constant A. The kinetic term is unaffected by this field redefinition (change of variables). But the potential term changes,

$$V(\varphi) = a_0' + a_1'\, \varphi + a_2'\, \varphi^2 + a_3'\, \varphi^3 + a_4\, \varphi^4,$$

where the new coefficients are,

$$\begin{aligned} a_0' &= a_0 - a_1 A + a_2 A^2 - a_3 A^3 + a_4 A^4, \\ a_1' &= a_1 - 2a_2 A + 3a_3 A^2 - 4a_4 A^3, \\ a_2' &= a_2 - 3a_3 A + 6a_4 A^2, \\ a_3' &= a_3 - 4a_4 A. \end{aligned}$$

Again we can forget about the a_0' because it is an overall constant piece in the lagrangian. Next we can choose A such that it satisfies the equation,

$$a_1' = a_1 - 2a_2 A + 3a_3 A^2 - 4a_4 A^3 = 0,$$

Then the linear term disappears, and we are left with the form of the potential,

$$V(\varphi) = a_2' \varphi^2 + a_3' \varphi^3 + a_4' \varphi^4.$$

We will just drop the primes because these are unspecified constants anyways, and work with the following restricted form of the potential,

$$V(\varphi) = a_2\, \varphi^2 + a_3\, \varphi^3 + a_4\, \varphi^4. \tag{8}$$

3.1 Some dimensional analysis

Let's determine the (mass) dimensions of the field, φ, the constants in the potential, a_2, a_4. Before doing that recall that we are using natural units in which $\hbar = c = 1$, both are dimensionless $[\hbar] = [c] = 0$. Also recall that mass has dimensions, $[M] = 1$, and thus in natural units, length has negative dimensions, $[L] = -1$ while [Linear Momentum] = [Energy] = 1. The action is dimensionless $[I] = 0$. Now, to determine the dimensions of φ, we look at the kinetic term in the action,

$$I = \int d^4x \, \frac{1}{2}\partial_\mu\varphi \, \partial^\mu\varphi + \ldots$$

Since dimensions of both sides must match we look at dimensions of both sides. The dimensions of the LHS of the above dimensional equation is zero. This means

$$\begin{aligned} 0 = [I] &= \left[d^4x \, \frac{1}{2}\partial_\mu\varphi \, \partial^\mu\varphi\right] \\ &= \left[d^4x\right] + \left[\partial_\mu\varphi \, \partial^\mu\varphi\right] \\ &= 4\,[L] + 2\,[\partial_\mu\varphi] \\ &= \left[L^4\right] + 2\left[\frac{\varphi}{L}\right] \\ &= 4\,[L] + 2\,[\varphi] - 2\,[L] \\ &= -2 + 2\,[\varphi] \end{aligned}$$

which gives,

$$[\varphi] = 1,$$

i.e. it has same dimensions as mass. These naive dimensions we are determining based on dimensional analysis are referred to the classical dimensions or *engineering dimensions* of φ.

Having determined the dimension of the field φ, it will now be easy to determine the engineering dimensions of the constants, m and λ. Again we start from the fact that the action is dimensionless, so the dimension of each term from the potential energy integral should vanish as well,

$$\left[\int d^4x \; a_2 \, \varphi^2\right] = 0, \qquad \left[\int d^4x \; a_4 \, \varphi^4\right] = 0.$$

These two conditions immediately imply,

$$[a_2] = 2 = \left[M^2\right],$$

$$[a_3] = 1, \quad [a_4] = 0.$$

Since the constant, a_2 has dimensions of mass squared we will rechristen the coefficients, $a_2 = \frac{m^2}{2}$, $a_3 = \frac{g_3}{3!}$and $a_4 = \frac{\lambda}{4!}$, to make the potential look consistent with the literature,

$$V(\varphi) = \frac{m^2}{2}\,\varphi^2 + \frac{g_3}{3!}\varphi^3 + \frac{\lambda}{4!}\,\varphi^4.$$

The quadratic piece, $\frac{m^2}{2}\,\varphi^2$ is referred to as the "*mass term*" and the cubic and quartic pieces, namely $\frac{g_3}{3!}\varphi^3$ and $\frac{\lambda}{4!}\,\varphi^4$ are referred to as the "*(self) coupling terms*" or "*(self) interaction terms*"[1].

[1] The quartic coupling constant, λ is dimensionless (in $3+1$ spacetime dimensions only. In higher or lower dimensions it will have non-zero dimensions).

The nomenclature will become obvious in the next few sections. It will turn out this restriction to quadratic and quartic potentials, i.e. setting $g_3 = 0$, will cover a large class of physical systems especially in particle physics and statistical mechanics and is a widely studied system which goes by the name of "$\lambda\varphi^4$ **theory**". Terms with higher powers of φ, are also undesirable in a fundamental theory since they will lead to quantum theory with no predictive power (non-renormalizable), so we will not consider them. However, for effective field theories such nonrenormalizable terms are fine.

The equation of motion for the λ-φ^4 theory is,

$$\Box\varphi + m^2\varphi + \frac{\lambda}{3!}\varphi^3 = 0,$$

or as is more customarily presented in books,

$$\left(\Box + m^2\right)\varphi = -\frac{\lambda}{3!}\varphi^3.$$

Now the significance of calling the constant λ, a coupling is obvious. In the absence of this term, one has a linear equation, namely the Klein-Gordon equation,

$$\left(\Box + m^2\right)\varphi = 0.$$

This can be thought of as a free field because a linear system of partial differential equations has solutions which obey the superposition principle, namely, if φ_1 and φ_2 are two independent solutions, then the linear combination, $\varphi = c_1\varphi_1 + c_2\varphi_2$ is a solution as well. Physically this means the two waves/disturbances, φ_1 and φ_2 do not see other and "pass thru" each other unaffected. That is why despite the fact that the term $m^2\varphi^2$ appears in the potential energy, it is not thought of as a coupling or interaction since it does not lead to any real physical interactions (say between the solutions, φ_1 and φ_2). On the contrary, introduction of the $\lambda\varphi^4$ term renders the equation of motion for the field nonlinear and which means, one cannot have a linear superposition, $\varphi = c_1\varphi_1 + c_2\varphi_2$ to be a solution if φ_1 and φ_2 are two independent solutions. Physically speaking, the two waves/disturbances, φ_1 and φ_2, cannot pass through each other unaffected in the presence of the λ-term. Hence we are justified in calling λ a coupling constant and the associated term, $\lambda\varphi^4$, the coupling/interaction term.

4 Free scalar field theory: Klein-Gordon equation

The free scalar field is described by the Klein-Gordon equation, namely,

$$\left(\Box + m^2\right)\varphi(x) = 0. \tag{9}$$

This equation is obtained as the Euler-Lagrange equation from varying the action,

$$I\left[\varphi\right] = \int d^4x \, \left(\frac{1}{2}\partial_\mu\varphi \, \partial^\mu\varphi - \frac{m^2}{2}\,\varphi^2\right).$$

Here we solve the Klein-Gordon equation. First note that if we set the parameter, $m = 0$, then the Klein-Gordon equation reduces to the wave equation, $\Box\varphi = 0$, which naturally will admit (plane) wave solutions. Anticipating a similar result for the Klein-Gordon equation we will first expand the field in a plane-wave basis, aka a four dimensional Fourier transform,

$$\varphi(x) = \int \frac{d^4x}{(2\pi)^4}\, \tilde{\varphi}(k)\, e^{-ik.x}. \tag{10}$$

Here, $k.x = k_\mu x^\mu$, and the four-dimensional wave-vector, k is defined by the components, $k^\mu = (c, \boldsymbol{k})$ in natural units. Since $x^\mu = (t, \boldsymbol{x})$, we have $k.x = \omega t - \boldsymbol{k}\cdot\boldsymbol{x}$. From now on we will use natural units and omit all factors of speed of light, c. Since we are restricting ourselves to ***real*** scalar fields,

$$\varphi^*(x) = \varphi(x),$$

which implies one needs to impose the condition,

$$\tilde{\varphi}(-k) = \tilde{\varphi}^*(k).$$

Substituting this plane-wave basis expansion of $\varphi(x)$ in the Klein-Gordon equation, one gets,

$$\left(\Box + m^2\right) \int \frac{d^4x}{(2\pi)^4}\, \tilde{\varphi}(k)\, e^{-ik.x} = 0.$$

Since the only dependence on x is contained in the phase factor, one has,

$$\left(\Box + m^2\right) \int \frac{d^4x}{(2\pi)^4}\, \tilde{\varphi}(k)\, e^{-ik.x} = \int \frac{d^4x}{(2\pi)^4}\, \tilde{\varphi}(k)\, \left(\Box + m^2\right) e^{-ik.x}.$$

Next note that, $\partial_\mu(e^{-ik.x}) = -ik_\mu e^{-ik.x}$, i.e. a derivative acting on the phase-factor pulls down a factor of $-ik$ from the exponent, and one thus one has,

$$\int \frac{d^4x}{(2\pi)^4}\, \tilde{\varphi}(k)\, \left(\Box + m^2\right) e^{-ik.x} = \int \frac{d^4x}{(2\pi)^4}\, \tilde{\varphi}(k)\, \left(-k^2 + m^2\right) e^{-ik.x}.$$

So the Klein-Gordon equation becomes,

$$\int \frac{d^4x}{(2\pi)^4}\, \tilde{\varphi}(k)\, \left(k^2 - m^2\right) e^{-ik.x} = 0.$$

Since the $e^{-ik.x}$ for different $k \in \mathbb{R}^4$ constitute a basis, for this equation to be valid one must have all basis coefficients vanishing, i.e.,

$$\tilde{\varphi}(k)\, \left(k^2 - m^2\right) = 0.$$

Next recall $k^2 = \omega^2 - \boldsymbol{k}^2$, where $\boldsymbol{k}^2 = \boldsymbol{k}\cdot\boldsymbol{k}$ and using it one can rewrite the above equation as,

$$\tilde{\varphi}(k)\, \left(\omega^2 - \omega_{\boldsymbol{k}}^2\right) = 0, \tag{11}$$

where we have defined $\omega_{\boldsymbol{k}} = \sqrt{\boldsymbol{k}^2 + m^2}$. This equation can be thought of as the momentum space (or Fourier space) version of Klein-Gordon equation. The solution to this equation, which is now an algebraic equation, is very simple,

$$\tilde{\varphi}(k) \begin{cases} = 0, \omega^2 \neq \omega_{\boldsymbol{k}}^2 \\ \neq 0, \omega^2 = \omega_{\boldsymbol{k}}^2. \end{cases}$$

A better way of rewriting this is by means of the Dirac delta function,

$$\tilde{\varphi}(k) = \varphi(k)\ \delta(\omega^2 - \omega_{\boldsymbol{k}}^2).$$

The delta function can be further simplified using the identity,

$$\delta\left(f(x)\right) = \sum_i \frac{\delta(x - x_i)}{|f'(x_i)|}$$

where the x_i's are the roots of the equation, $f(x) = 0$. This implies,

$$\delta(\omega^2 - \omega_{\boldsymbol{k}}^2) = \frac{1}{2\omega_{\boldsymbol{k}}}\delta(\omega - \omega_{\boldsymbol{k}}) + \frac{1}{2\omega_{\boldsymbol{k}}}\delta(\omega + \omega_{\boldsymbol{k}}),$$

and,

$$\tilde{\varphi}(k) = \varphi(k)\ \delta(\omega^2 - \omega_{\boldsymbol{k}}^2) = \frac{1}{2\omega_{\boldsymbol{k}}}\varphi(k)\ \left(\delta(\omega - \omega_{\boldsymbol{k}}) + \delta(\omega + \omega_{\boldsymbol{k}})\right).$$

Plugging this back in the mode-expansion (10), one gets the general solution,

$$\varphi(x) = \int \frac{d^4k}{(2\pi)^4}\ \frac{1}{2\omega_{\boldsymbol{k}}}\varphi(k)\ \left(\delta(\omega - \omega_{\boldsymbol{k}}) + \delta(\omega + \omega_{\boldsymbol{k}})\right) e^{-ik.x}.$$

Next recall that, $d^4k = d\omega\ d^3\boldsymbol{k}$, and we can easily do the ω integral due to the delta function(s). Performing the ω-integral for both terms gives us the form of the general solution to be,

$$\varphi(x) = \int \frac{d^3\boldsymbol{k}}{(2\pi)^2}\ \frac{1}{(2\pi)\,(2\omega_{\boldsymbol{k}})}\varphi(\omega_{\boldsymbol{k}}, \boldsymbol{k})\ e^{-i(\omega_{\boldsymbol{k}}t - \boldsymbol{k}\cdot\boldsymbol{x})} + \int \frac{d^3\boldsymbol{k}}{(2\pi)^2}\ \frac{1}{(2\pi)\,(2\omega_{\boldsymbol{k}})}\varphi(-\omega_{\boldsymbol{k}}, \boldsymbol{k})\ e^{i(\omega_{\boldsymbol{k}}t + \boldsymbol{k}\cdot\boldsymbol{x})}.$$

Now we make a change of variable in the second term (integral), namely, $\boldsymbol{k} \to -\boldsymbol{k}$. Since $\omega_{\boldsymbol{k}}$ is an even function of $\boldsymbol{k}$, it is unaffected. However, $\tilde{\varphi}(-\omega_{\boldsymbol{k}}, \boldsymbol{k}) \to \tilde{\varphi}(-\omega_{\boldsymbol{k}}, -\boldsymbol{k})$ and $\boldsymbol{k}\cdot\boldsymbol{x} \to -\boldsymbol{k}\cdot\boldsymbol{x}$. Making these changes the general solution now looks like,

$$\varphi(x) = \int \frac{d^3\boldsymbol{k}}{(2\pi)^2}\ \frac{1}{(2\pi)\,(2\omega_{\boldsymbol{k}})}\left[\varphi(\omega_{\boldsymbol{k}}, \boldsymbol{k})\ e^{-i(\omega_{\boldsymbol{k}}t - \boldsymbol{k}\cdot\boldsymbol{x})} + \varphi(-\omega_{\boldsymbol{k}}, -\boldsymbol{k})\ e^{i(\omega_{\boldsymbol{k}}t - \boldsymbol{k}\cdot\boldsymbol{x})}\right].$$

This is certainly real valued once we recall that,

$$\varphi(-\omega_{\boldsymbol{k}}, -\boldsymbol{k}) = \varphi^*(\omega_{\boldsymbol{k}}, \boldsymbol{k}).$$

Thus, we can write the final form of the most general solution of the Klein-Gordon equation in a plane wave basis to be,

$$\varphi(x) = \int \frac{d^3\boldsymbol{k}}{(2\pi)^2}\ \frac{1}{(2\pi)\,(2\omega_{\boldsymbol{k}})}\varphi(\omega_{\boldsymbol{k}}, \boldsymbol{k})\ e^{-i(\omega_{\boldsymbol{k}}t - \boldsymbol{k}\cdot\boldsymbol{x})} + \text{cc} \tag{12}$$

where "cc" stands for complex conjugate. **Note that the coefficients, $\varphi(\omega_{\boldsymbol{k}}, \boldsymbol{k})$ are completely arbitrary**.

Thus the solutions to KG are indeed plane waves albeit with a dispersion relation,

$$\omega_{\boldsymbol{k}} = \sqrt{\boldsymbol{k}^2 + m^2}.$$

The phase velocity is,

$$c_{\boldsymbol{k}} = \frac{\omega_{\boldsymbol{k}}}{|\boldsymbol{k}|} = \sqrt{1 + \frac{m^2}{\boldsymbol{k}^2}}.$$

Thus these are dispersive waves, even in vacuo - different frequencies travel with different phase velocities. One might be alarmed by the fact that, $c_{\boldsymbol{k}} > 1$, i.e. phases travel faster than light and hence violate causality. However one must recall that while phases can travel faster than the speed of light, the whole wave-packet moves with the *group velocity*. We can compute the group velocity of the KG wave-packet,

$$v_g = \frac{d\omega_{\boldsymbol{k}}}{d\,|\boldsymbol{k}|} = \frac{|\boldsymbol{k}|}{\sqrt{\boldsymbol{k}^2 + m^2}}.$$

Thus $v_g < 1$ and information (energy-momentum) *does not* travel faster than light and there is no violation of causality!

Homework: Find the dimensions of the real scalar field in D spacetime dimensions. Find the dimensions of the cubic coefficient g_3 when $D = 6$.