Spring 2022: Quantum Field Theory (PH 6418/ EP 4618)

Notes for lecture 11*

April 4, 2022

1 Global Continuous Symmetries: Noether's (first) theorem & construction of charges

Noether's first theorem states that whenever a physical system has an *continuous global* symmetry i.e. when the the action functional of the system is invariant under some transformation rules of the coordinates and/or configuration space variables and the symmetry transformation parameter takes on values continuously on the real line or some subset of the real line (continuous) and the parameter remains same at all points in spacetime (global), then there exists a conserved charge corresponding to that symmetry. In this section, we will apply Noether's algorithm to construct the conserved charges for the free scalar system. First we will look at spacetime symmetries such as Lorentz and translation symmetries. Since the analysis of spacetime symmetries is virtually identical for real and complex scalar field theories, we will be content to consider the *real* scalar field theory. The theory of the real scalar field, $\varphi(x)$ can be described by the action,

$$I\left[\varphi(x)\right] = \int d^4x\, \mathcal{L},$$

where the Lagrangian $\mathcal{L}$ is a function of the scalar, $\varphi(x)$ and it's spacetime derivatives $\partial_\mu\varphi(x)$,

$$\mathcal{L} = \mathcal{L}\left(\varphi(x), \partial_\mu\varphi(x)\right) = \mathcal{L} = \frac{1}{2}\partial_\mu\varphi\, \partial^\mu\varphi - V(\varphi(x))$$

The integration range is over all space and time. In particular, for the **free scalar,** the Lagrangian can be taken to be,

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\, \partial^\mu\varphi - \frac{1}{2}m^2\varphi^2. \tag{1}$$

This form is dictated by Lorentz invariance i.e. the Lagrangian density **must be Lorentz scalar**. m is a Lorentz invariant quantity with the dimensions of mass (or energy). (Upon quantizing the system, the parameter, m will turn out to be the mass of the scalar field quanta/particles).

The symmetries/invariances of the real scalar field action are:

- Lorentz invariance $x \to x' = \Lambda x$, $\varphi(x) \to \varphi'(x') = \varphi(x)$.

*Typos and errors should be emailed to the Instructor: Shubho Roy (email: sroy@phy.iith.ac.in)

- Translation invariance $x \to x' = x + a$, $\varphi(x) \to \varphi'(x') = \varphi(x)$.
- Discrete internal symmetry, $\varphi \to \varphi' = -\varphi$. (Only if the lagrangian contains **even powers** of φ).

Checks:

- Lorentz invariance is rather obvious because most terms in the action is Lorentz invariant, d^4x, m^2, φ^2. Even the kinetic term, $\partial_\mu\varphi\, \partial^\mu\varphi$, is because the Lorentz index μ is contracted, vis:

$$\partial_\mu\varphi(x) \to \partial'_\mu\varphi'(x') = \Lambda_\mu{}^\nu\, \partial_\nu\varphi(x),$$

$$\begin{aligned}
\partial_\mu\varphi(x)\, \partial^\mu\varphi(x) \to \partial'_\mu\varphi'(x')\, \partial'^\mu\varphi'(x') &= \Lambda_\mu{}^\nu\, \partial_\nu\varphi(x)\, \Lambda^\mu{}_\alpha \partial^\alpha\varphi(x) \\
&= (\Lambda_\mu{}^\nu)\ (\Lambda^\mu{}_\alpha)\ \partial_\nu\varphi(x)\, \partial^\alpha\varphi(x) \\
&= \delta^\nu_\alpha\, \partial_\nu\varphi(x)\, \partial^\alpha\varphi(x) \\
&= \partial_\nu\varphi(x)\, \partial^\nu\varphi(x).
\end{aligned}$$

Thus the action

$$\begin{aligned}
I\left[\varphi'(x')\right] &= \int d^4x' \left[\frac{1}{2}\partial'_\mu\varphi'(x')\partial'^\mu\varphi'(x') - \frac{1}{2}m^2\varphi'^2(x)\right] \\
&= \int d^4x \left[\frac{1}{2}\partial_\mu\varphi(x)\partial^\mu\varphi(x) - \frac{1}{2}m^2\varphi^2(x)\right] \\
&= I\left[\varphi(x)\right],
\end{aligned}$$

remains invariant.

- Translation invariance is also obvious because the action integral being defined over all space and time i.e ranges of integration being $(-\infty, \infty)$, is independent of the origin of coordinates and there is no *explicit* dependence on the coordinates, x. Recall that under translations, namely,

$$x \to x' = x + a,$$

the field φ transforms as,

$$\varphi'(x') = \varphi(x),$$

What about the kinetic piece containing terms such as $\partial_\mu\varphi$. Such a term seems to care about the spacetime coordinate through the derivative, $\partial_\mu = \frac{\partial}{\partial x^\mu}$. Actually even this derivative is independent of the shift in origin because under a shift of origin of coordinates,

$$x \to x' = x + a,$$

Conversely,

$$x = x' - a$$

the derivative transforms as

$$\begin{aligned}\frac{\partial}{\partial x^\mu} \to \frac{\partial}{\partial x'^\mu} &= \frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial}{\partial x^\nu}, \\ &= \frac{\partial\left(x'^\nu - a^\nu\right)}{\partial x'^\mu}\frac{\partial}{\partial x^\nu} \\ &= \frac{\partial x'^\nu}{\partial x'^\mu}\frac{\partial}{\partial x^\nu} \\ &= \partial^\nu_\mu \frac{\partial}{\partial x^\nu} \\ &= \frac{\partial}{\partial x^\mu}.\end{aligned}$$

So the derivative remains unchanged. (Here since a is a constant its derivative vanishes, and we get $\frac{\partial(x'^\nu - a^\nu)}{\partial x'^\mu} = \frac{\partial x'^\nu}{\partial x'^\mu}$).

- Discrete internal symmetry such as $\varphi \to \varphi' = -\varphi$ is also obvious when the Lagrangian contains even powers of φ. However as they are discrete (non-continuous) and do not obey Noether's theorem and they will not give rise to conserved charges.

Next consider the complex scalar field theory, defined by the action,

$$I\left[\Phi(x), \Phi^\dagger(x)\right] = \int d^4x \left[(\partial_\mu\Phi)^\dagger \partial^\mu\Phi - V\left(\Phi^\dagger\Phi\right)\right]. \tag{2}$$

This action is symmetric under:

- Lorentz symmetry: The action is (2) is invariant under restricted Lorentz transformations, i.e. boosts and rotations

$$\begin{aligned} x \to x' &= \Lambda\, x, \\ \Phi(x) \to \Phi'(x') &= \Phi(x). \end{aligned}$$

This can be concluded from the fact that each term in the Lagrangian density (2) is a Lorentz invariant (scalar). This is a continuous spacetime symmetry, i.e. the symmetry transformations are parameterized by 6 real parameters, namely 3 angles (rotations) and 3 rapidities (boosts),

$$\Lambda = \Lambda\left(\theta_1, \theta_2, \theta_3, \eta_1, \eta_2, \eta_3\right).$$

Since the symmetry parameters, angles and rapidities are same at all space time points, this is also a global symmetry.

- Translation symmetry: The action (2) is invariant under,

$$\begin{aligned} x \to x' &= x - a, \\ \Phi(x) \to \Phi'(x') &= \Phi(x). \end{aligned}$$

This can be proven exactly on the same lines as the real scalar. Again this is a continuous global symmetry depending on 4 real parameters, namely the 4 shift variables, a^μs and these shift parameters are same at all spacetime points.

- Complex Conjugation (or charge conjugation) symmetry: The complex scalar action is invariant under the switch where the field is replaced by its complex conjugate and vice versa,

$$\Phi(x) \to \Phi'(x) = \Phi^\dagger(x).$$

This charge conjugation symmetry is a discrete global symmetry. This is a discrete symmetry because in the symmetry transformation, no real continuous parameters appear. This is also a global symmetry because the symmetry transformation equation holds for all x i.e. identical at all spacetime points.

- $U(1)$ or Phase transformation symmetry: As we have noted earlier complex scalar field theory action Eq. (2) is invariant under multiplication by a **constant** complex phase factor $e^{i\,\alpha}$,

$$\begin{aligned} \Phi \to \Phi' &= e^{-i\,\alpha}\Phi, \\ \Phi^\dagger \to \Phi'^\dagger &= e^{i\,\alpha}\Phi^\dagger. \end{aligned} \tag{3}$$

The phase,α is necessarily a real number. Since a complex phase is unitary 1×1 matrix i.e. the complex conjugation is also the inverse,

$$\left(e^{-i\,\alpha}\right)^\dagger = \left(e^{-i\,\alpha}\right)^{-1},$$

such phases are also called $U(1)$ factors (U stands for Unitary matrix and since a number is a 1×1 matrix, $U(1)$ is unitary matrix of size 1×1). Since this symmetry transformation does not touch spacetime but only changes the fields, such a symmetry is called an **internal symmetry**. Also note that since α is a constant i.e. not a function of spacetime, it is a **global** symmetry (**global = same everywhere = independent of spacetime location**).

Check: Under the U(1) symmetry Eq. (3), the combination $\Phi^\dagger\Phi$ is obviously invariant,

$$\begin{aligned} \Phi'^\dagger\Phi' &= \left(e^{i\,\alpha}\Phi^\dagger\right)\,\left(e^{-i\,\alpha}\Phi\right) \\ &= \Phi^\dagger\Phi. \end{aligned}$$

This implies any function of the product $\Phi^\dagger\Phi$ is also invariant.

$$V\left(\Phi'^\dagger\Phi'\right) = V\left(\Phi^\dagger\Phi\right).$$

Note that this is true whether α is a constant or a function of spacetime i.e. $\alpha(x)$. Next let's look at the kinetic term,

$$\begin{aligned} \left(\partial_\mu\Phi^\dagger\right)\left(\partial^\mu\Phi\right) \to \left(\partial_\mu\Phi'^\dagger\right)\left(\partial^\mu\Phi'\right) &= \partial_\mu\left(e^{i\,\alpha}\Phi^\dagger\right)\partial^\mu\left(e^{-i\,\alpha}\Phi\right), \\ &= e^{i\,\alpha}\left(\partial_\mu\Phi^\dagger\right)\; e^{-i\,\alpha}\left(\partial^\mu\Phi\right) \\ &= \left(\partial_\mu\Phi^\dagger\right)\left(\partial^\mu\Phi\right). \end{aligned}$$

So this kinetic term in the action is also invariant because α is a constant and the derivative does not act on it. If α was a function of spacetime, $\alpha = \alpha(x)$, the derivative would have acted on it and the term would not be invariant. Incidentally, a spacetime dependent phase $\alpha(x)$ is called a **local** $U(1)$ **transformation.**

2 Noether algorithm to extract conserved charges from global continuous symmetries

In the case of spacetime symmetries such as Lorentz transformations and translations, we see that the parameters $\Lambda^{\mu}{}_{\nu}$ and a^{μ} are indeed constants and not functions of spacetime i.e. these are **global** symmetries. So Noether's theorem applies in these cases and tells us that there must be conserved charges. Here we obtain the expressions for those charges using the so called ***"Noether algorithm"***. In a nutshell, the algorithm for extracting Noether charges is as follows:

1. First make the global symmetry parameter, say ε, infinitesimal. This is allowed because the symmetry parameter takes values on some continuous segment of the real line which *includes the origin*. In case of translations we will take the shift parameter, a^{μ} to be infinitesimal, and in case of rotations the angle of rotation, θ is to be taken infinitesimal. When this done, in all subsequent steps of the procedure we will only keep terms which are up to $O(\varepsilon)$, i.e. linear order in the infinitesimal symmetry parameter. Higher order terms i.e. $O\left(\varepsilon^2\right)$ will be dropped. Note that ε parameter can be a scalar or a tensor of any rank, i.e. it might have tensor indices which we are suppressing at the moment. E.g. in case of Lorentz transformations, the infinitesimal form is

$$\Lambda^{\mu}{}_{\nu} = \delta^{\mu}_{\nu} + \omega^{\mu}{}_{\nu},$$

   the symmetry transformation parameters are rank $(1,1)$ tensors, while for translations, the infinitesimal symmetry parameter is,

$$a^{\mu}$$

   which is a rank $(1,0)$ tensor. Here we denote a generic infinitesimal symmetry transformation parameter by ε without displaying its tensor indices.

2. Next we will ***temporarily assume*** that the symmetry parameter ε, is a function of spacetime, i.e. $\varepsilon = \varepsilon(x)$. E.g., in case of Lorentz transformations, we will temporarily make $\Lambda^{\mu}{}_{\nu} = \Lambda^{\mu}{}_{\nu}(x) = \delta^{\mu}_{\nu} + \omega^{\mu}{}_{\nu}(x)$ and in the case of translations $a^{\mu} = a^{\mu}(x)$.

3. **Computing the change in the action:** In step 3, we compute the change in the action integral to first order in the parameter $\varepsilon(x)$, which should be,

$$\delta I = -\int d^4x \ \left(\partial_{\mu}\varepsilon(x)\right)\ j^{\mu} + O\left(\varepsilon^2\right).$$

   For example, for translations one must have the change in action,

$$\delta I = -\int d^4x \ \left(\partial_{\mu}a_{\nu}(x)\right)\ \theta^{\mu\nu} + O\left(a^2\right),$$

   while for Lorenz transformations we must have,

$$\delta I = -\int d^4x \ \left(\partial_{\lambda}\omega_{\mu\nu}(x)\right)\ M^{\lambda\,\mu\nu} + O\left(\omega^2\right).$$

   This form is consistent with the our expectation for a global symmetry. Right now, the symmetry parameter is not global since we have temporarily assumed it (them) to be functions of spacetime, and this is ***NOT*** a symmetry of the action, the hence action must have a non-vanishing change. However, in the special case when a and ω are constants these expressions must vanish as the action is supposed to invariant under the global/constant changes which is a symmetry of the system.

4. **Integration by parts and discarding surface terms:** From the expression for the δI, read off the companion coefficient term, j^μ which are some functions of the field(s) and its (their) derivatives. These are the conserved Noether currents! They will obey a continuity type equation when the equations of motion hold. This can be inferred from the above expressions for δI by a simple integration by parts and abandoning the total derivative term (we can abandon this term under the assumption that the surface term goes to zero at infinity),

$$\begin{aligned}
\delta I &= -\int d^4x \; (\partial_\mu \varepsilon(x)) \; j^\mu \\
&= -\int d^4x \; \partial_\mu \left(\varepsilon(x) \; j^\mu\right) + \int d^4x \; \varepsilon(x) \, \partial_\mu j^\mu \\
&= -\cancelto{0}{\int d^3S_\infty \; n_\mu \left(\varepsilon(x) \; j^\mu\right)} + \int d^4x \; \varepsilon(x) \, \partial_\mu j^\mu \\
&= \int d^4x \; \varepsilon(x) \, \partial_\mu j^\mu .
\end{aligned}$$

5. **Go on shell**: Next we Use the variational principle to demand the change in the action to vanish around classical configurations (whereby the equation of motion holds). The change of the action might not vanish when the symmetry parameter is turned local (function of spacetime) but now since we are talking about configurations around the equation of motion, the variation of the action has to vanish for *arbitrary* variations, *including* the case when the variation happens to be with the symmetry parameter being local. Thus when equations of motion hold, one has

$$\delta I = 0,$$

or,

$$\int d^4x \; \varepsilon(x) \, \partial_\mu j^\mu = 0$$

The only way this integral can vanish for arbitrary function $\varepsilon(x)$ is when rest of the integrand vanishes, i.e.

$$\partial_\mu j^\mu = 0.$$

This is nothing but the continuity equation for a current density, j^μ! We know from past experience that it represents a conservation law. More concretely say for translations, one has when the equation of motion holds,. a^ν becomes a constant, it can be pulled out of the integral and we have the expression,

$$\delta I = a^\nu \int d^4x \; a_\nu(x) \; (\partial_\mu \theta^{\mu\nu}) = 0.$$

which immediately leads to,

$$\partial_\mu \theta^{\mu\nu} = 0,$$

the continuity equation for a current density. This current density corresponding to translation/shift symmetry of a field theory is called the *canonical stress-energy-momentum* tensor, $\theta^{\mu\nu}$. One can explicitly check, using the equations of motion of the fields, that the above continuity equation indeed holds.

6. Construct the Noether charge by performing the spatial volume integral

$$Q = \int d^3\mathbf{x}\, j^0.$$

where j^μ is a conserved Noether current-density. This follows easily from the continuity equation.

$$\partial_\mu j^\mu = 0 \Rightarrow \frac{\partial j^0}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{j} = 0.$$

Which implies,

$$\frac{\partial j^0}{\partial t} = -\boldsymbol{\nabla} \cdot \mathbf{j}.$$

Taking a (spatial) volume integral of both sides,

$$\int d^3\mathbf{x}\, \frac{\partial j^0}{\partial t} = -\int d^3\mathbf{x}\, \left(\boldsymbol{\nabla} \cdot \mathbf{j}\right).$$

Now in the LHS we swap the space-integral and time derivative, $\int d^3\mathbf{x}\, \frac{\partial j^0}{\partial t} = \frac{d}{dt}\left(\int d^3\mathbf{x}\, j^0\right)$, one the RHS we convert it to a surface integral at spatial infinity using Gauss divergence theorem,

$$\frac{d}{dt}\left(\int d^3\mathbf{x}\, j^0\right) = -\int_{S^\infty} dS\, \hat{\mathbf{n}} \cdot \mathbf{j}.$$

Using the right boundary conditions the surface term at spatial infinity on the RHS vanishes and we leads to the conservation law,

$$\frac{d}{dt}\left(\underbrace{\int d^3\mathbf{x}\, j^0}_{=Q}\right) = 0.$$

For example, for the case of translations, the conserved Noether charge is,

$$P_\nu = \int d^3\mathbf{x}\, \theta^0{}_\nu,$$

which are nothing but the linear momentum 4-vector.

7. Optional but highly recommended step: Check that the charge is conserved by using the equation of motion,

$$\frac{dQ}{dt} = 0.$$

In practice we will simply do up to step 4 and read off the current, $J^{\mu\cdots}$, and skip step 5 and perform step 6, i.e. integrate $J^{0\cdots}$ to extract the Noether charge.

3 Demonstration of the Noether algorithm: Noether charge for the global $U(1)$ symmetry of the complex field theory

Here we obtain the conserved charge for the global $U(1)$ symmetry of the complex scalar field using the Noether method. For simplicity we work with the free theory,

$$V\left(\Phi^\dagger\Phi\right) = m^2\Phi^\dagger\Phi.$$

1. As a first step in the process, one expresses the $U(1)$ symmetry transformation of the field (and its complex conjugate) in infinitesimal form,

$$\begin{aligned}\Phi \to \Phi' &= e^{-i\,\alpha}\Phi \\ &= (1 - i\alpha + O(\alpha^2))\Phi \\ &\approx \Phi - i\alpha\Phi,\end{aligned}$$

   while the complex conjugate field (to first order in α) changes to,

$$\Phi'^\dagger \approx \Phi^\dagger + i\alpha\Phi^\dagger.$$

2. Next step in the process is to ***temporarily assume***, α is a function of spacetime, i.e. $\alpha = \alpha(x)$,

$$\Phi \to \Phi' = \Phi - i\alpha(x)\,\Phi,$$
$$\Phi^\dagger \to \Phi'^\dagger = \Phi^\dagger + i\alpha(x)\,\Phi^\dagger.$$

   Since $\alpha = \alpha(x)$ is not a symmetry of the action, the action will change if we replace, $\Phi \to \Phi' = \Phi - i\alpha(x)\,\Phi$, in the action (2) i.e. $I\left[\Phi', \Phi'^\dagger\right] \neq I\left[\Phi, \Phi^\dagger\right]$.

3. The next step in the Noether method is to compute the change in the action, $\delta I = I\left[\Phi', \Phi'^\dagger\right] - I\left[\Phi, \Phi^\dagger\right]$. For that we first need to find the change in the derivative of the field,

$$\begin{aligned}\partial^\mu\Phi \to \partial^\mu\Phi' &= \partial^\mu\left(\Phi - i\alpha(x)\,\Phi\right) \\ &= \partial^\mu\Phi - i\left(\partial^\mu\alpha\right)\Phi - i\alpha(x)\,\partial^\mu\Phi,\end{aligned}$$

   and the derivative of the complex conjugate,

$$\begin{aligned}\partial_\mu\Phi^\dagger \to \partial_\mu\Phi'^\dagger &= \partial_\mu\left(\Phi^\dagger + i\alpha(x)\Phi^\dagger\right) \\ &= \partial_\mu\Phi^\dagger + i\left(\partial_\mu\alpha\right)\Phi^\dagger + i\alpha(x)\,\partial_\mu\Phi^\dagger.\end{aligned}$$

   Using these expressions, the action for the transformed fields is,

$$\begin{aligned}I\left[\Phi', \Phi'^\dagger\right] &= \int d^4x\,\left[\left(\partial_\mu\Phi'\right)^\dagger\partial^\mu\Phi' - m^2\Phi'^\dagger\Phi'\right] \\ &= \int d^4x\,\left[\partial_\mu\Phi^\dagger + i\left(\partial_\mu\alpha\right)\Phi^\dagger + i\alpha(x)\,\partial_\mu\Phi^\dagger\left(\partial^\mu\Phi - i\left(\partial^\mu\alpha\right)\Phi - i\alpha(x)\,\partial^\mu\Phi\right)\right. \\ &\qquad\left. - m^2\left(\Phi - i\alpha(x)\,\Phi\right)\left(\Phi^\dagger + i\alpha(x)\,\Phi^\dagger\right)\right] \\ &= \int d^4x\,\left[\left(\partial_\mu\Phi\right)^\dagger\partial^\mu\Phi - m^2\Phi^\dagger\Phi - i\partial_\mu\alpha\left(\Phi\partial_\mu\Phi^\dagger - \Phi^\dagger\partial^\mu\Phi\right)\right] + \cancelto{0}{O\left(\alpha^2\right)} \\ &= I\left[\Phi, \Phi^\dagger\right] - \int d^4x\,\partial_\mu\alpha\,i\,\left(\Phi\partial_\mu\Phi^\dagger - \Phi^\dagger\partial^\mu\Phi\right).\end{aligned}$$

So the first order (in α) change in the action is,

$$\begin{aligned} \delta I &= I\left[\Phi', \Phi'^{\dagger}\right] - I\left[\Phi, \Phi^{\dagger}\right] \\ &= -\int d^4x\, \partial_\mu \alpha(x)\, i\, \left(\Phi\partial_\mu\Phi^\dagger - \Phi^\dagger\partial^\mu\Phi\right). \end{aligned}$$

From this expression we can identify the conserved current corresponding to the global U(1) symmetry,

$$j^\mu = i\,\left(\Phi\partial_\mu\Phi^\dagger - \Phi^\dagger\partial^\mu\Phi\right). \tag{4}$$

One can easily check that is conserved on-shell (on-shell means the classical equation of motion holds). The equation of motion for the free complex scalar field (the Klein-Gordon equation) is

$$\left(\partial^2 + m^2\right)\Phi = \left(\partial^2 + m^2\right)\Phi^\dagger = 0. \tag{5}$$

Or,

$$\partial^2\Phi = -m^2\Phi, \qquad \partial^2\Phi^\dagger = -m^2\Phi^\dagger,$$

and we so have,

$$\begin{aligned} \partial_\mu j^\mu &= i\left(\Phi^\dagger\partial^2\Phi - \Phi\partial^2\Phi^\dagger\right) \\ &= i\left(-\Phi^\dagger m^2\Phi + \Phi m^2\Phi^\dagger\right) \\ &= 0. \end{aligned}$$

4. Finally, the conserved charge is then given by the volume integral,

$$Q = \int d^3\mathbf{x}\, j^0 = i\int d^3\mathbf{x}\,\left(\Phi\dot{\Phi}^\dagger - \Phi^\dagger\dot{\Phi}\right). \tag{6}$$

Check:

$$\begin{aligned} \frac{dQ}{dt} &= \frac{d}{dt} i\int d^3\mathbf{x}\,\left(\Phi\dot{\Phi}^\dagger - \Phi^\dagger\dot{\Phi}\right) \\ &= i\int d^3\mathbf{x}\,\frac{\partial}{\partial t}\left(\Phi\dot{\Phi}^\dagger - \Phi^\dagger\dot{\Phi}\right) \\ &= i\int d^3\mathbf{x}\,\left(\Phi\ddot{\Phi}^\dagger - \Phi^\dagger\ddot{\Phi}\right). \end{aligned}$$

Now we use the Klein-Gordon equation for the time-derivatives,

$$\begin{aligned} \ddot{\Phi} &= \left(\boldsymbol{\nabla}^2 - m^2\right)\Phi, \\ \ddot{\Phi}^\dagger &= \left(\boldsymbol{\nabla}^2 - m^2\right)\Phi^\dagger, \end{aligned}$$

and obtain,

$$\begin{aligned} \frac{dQ}{dt} &= i\int d^3\mathbf{x}\,\left(\Phi\boldsymbol{\nabla}^2\Phi^\dagger - \Phi^\dagger\boldsymbol{\nabla}^2\Phi\right) \\ &= i\int d^3\mathbf{x}\,\boldsymbol{\nabla}\cdot\left(\Phi\boldsymbol{\nabla}\Phi^\dagger - \Phi^\dagger\boldsymbol{\nabla}\Phi\right) \\ &= i\int_\infty d^2S\,\hat{\boldsymbol{n}}\cdot\left(\Phi\boldsymbol{\nabla}\Phi^\dagger - \Phi^\dagger\boldsymbol{\nabla}\Phi\right) \\ &= 0. \end{aligned}$$

Here in going from the penultimate to the ultimate step we used the boundary conditions at spatial infinity to set the current to zero,

$$\lim_{|\boldsymbol{x}|\to\infty} \Phi\boldsymbol{\nabla}\Phi^\dagger - \Phi^\dagger\boldsymbol{\nabla}\Phi = 0.$$

Alternatively if we have the current to be nonzero at spatial infinity we can make this term vanish by demanding that the integral of the current at surface at spatial infinity, i.e. net charge flux at infinity vanishes. This happens in magentostatics when you have an infinitely long current carrying wire - the net charge coming in from infinity is zero because the current coming in from infinity is equal to the current going out to infinity.

Homework: Check that the current (4) is conserved i.e. obeys the continuity equation $\partial_\mu j^\mu = 0$ not just for the free case i.e. when $V = m^2\Phi^\dagger\Phi$ but for a potential which is a more general function of $\Phi^\dagger\Phi$, i.e. $V(\Phi^\dagger\Phi)$. (Hint: Use the Euler-Lagrange equation of motion).

3.0.1 The charge conjugation symmetry of the complex scalar field theory flips the sign of U(1) charge

Recall that in addition to the continuous global $U(1)$ symmetry (3), there is another discrete internal symmetry of the complex field theory, namely, interchanging the field Φ with its complex conjugate, $\Phi^\dagger$,

$$\Phi \leftrightarrow \Phi^\dagger.$$

Under this symmetry of course the charge (polarity) of the scalar field also changes,

$$Q \to -Q.$$

This is why this discrete symmetry is dubbed as ***charge conjugation symmetry***. In the quantum theory, this discrete internal symmetry will transform particles to antiparticles and vice-versa.

Comment: This U(1) will soon be identified with the electric charge via coupling to the Maxwell field, A_μ.