Spring 2022: Quantum Field Theory (PH 6418/ EP 4618)

Notes for lecture 12*

April 7, 2022

1 Comment on Noether currents and charges

Consider a Noether current, j^μ obtained by following the Noether algorithm. It satisfies the continuity equation representing charge conservation,

$$\partial_\mu j^\mu = 0.$$

Now let $K^{\mu\nu}$ be an arbitrary antisymmetric tensor, i.e. $K^{\mu\nu} = -K^{\nu\mu}$. Then it follows that $J^\mu = j^\mu + \partial_\nu K^{\mu\nu}$ is also a conserved current. To wit,

$$\partial_\mu J^\mu = \underbrace{\partial_\mu j^\mu}_{=0} + \cancelto{0}{\partial_\mu \partial_\nu K^{\mu\nu}} = 0.$$

So the Noether current is *not unique* and is ambiguous up to addition by a term like $\partial_\nu K^{\mu\nu}$ where $K^{\mu\nu}$ is an arbitrary antisymmetric tensor. However the Noether charge is unique, provided $K^{0i} \sim \frac{1}{r^{2+}}$ as $r = |\boldsymbol{x}| \to \infty$. This can be shown as follows

$$\begin{aligned} Q' &= \int d^3\boldsymbol{x}\; J^0 \\ &= \underbrace{\int d^3\boldsymbol{x}\; j^0}_{=Q} + \int d^3\boldsymbol{x}\; \partial_\nu K^{0\nu} \\ &= Q + \int d^3\boldsymbol{x}\; \partial_i K^{0i} \\ &= Q + \cancelto{0}{\int_{r\to\infty} dS\; n_i\, K^{0i}} \\ &= Q. \end{aligned}$$

The second term vanishes because $dS \sim r^2$ while $K^{0i} \sim \frac{1}{r^{2+\delta}}$ as $r \to \infty$. So provided we impose suitable asymptotic fall offs on K^{0i}'s the Noether charge is unique.

*Typos and errors should be emailed to the Instructor: Shubho Roy (email: sroy@phy.iith.ac.in)

2 The canonical stress-energy-momentum tensor for the (real) scalar field theory

We use the Noether procedure to extract the stress-energy-momentum tensor for the (real) scalar field theory. First step is to turn the shift parameter infinitesimal, which in practice means that we will keep terms up to order one in a^μ. Then we set the infinitesimal symmetry parameter to not be constant (global), but instead a function of spacetime,

$$a^\mu = a^\mu(x).$$

So we have new coordinates,

$$x \to x'^\mu = x^\mu - a^\mu(x).$$

The Jacobian matrix components for the change of variables, $x \to x'$ are,

$$J^\mu{}_\nu = \frac{\partial x'^\mu}{\partial x^\nu} = \delta^\mu_\nu - \partial_\nu a^\mu,$$

i.e. it is a sum of the identity matrix and a small change, $\partial_\nu a^\mu$. Hence to first order of the shift, the Jacobian determinant is,

$$\begin{aligned} |J| &= 1 - \mathrm{trace}(\partial_\nu a^\mu) \\ &= 1 - \partial_\rho a^\rho. \end{aligned} \tag{1}$$

The derivatives transform like,

$$\begin{aligned} \partial_\mu \to \partial'_\mu &= \frac{\partial x^\nu}{\partial x'^\mu}\,\partial_\nu \\ &= \left(\delta^\nu_\mu + \partial_\mu a^\nu\right)\,\partial_\nu \\ &= \partial_\mu + \partial_\mu a^\nu\,\partial_\nu. \end{aligned} \tag{2}$$

Now let's look at the change in φ and it's derivatives, $\partial_\mu\varphi$. We have,

$$\varphi'(x') = \varphi(x),$$

while,

$$\begin{aligned} \partial'_\mu\varphi'(x') = \partial'_\mu\varphi(x) &= \frac{\partial x^\nu}{\partial x'^\mu}\,\partial_\nu\varphi(x) \\ &= \partial_\mu\varphi(x) + \partial_\mu a^\nu\,\partial_\nu\varphi(x). \end{aligned} \tag{3}$$

Now we are ready to compute the transformed action after this spacetime dependent translation,

$a(x)$ using the transformation equations (1-3) :

$$\begin{aligned}
I\left[\varphi'(x')\right] &= \int d^4x' \left[\frac{1}{2}\partial'_\mu\varphi'(x')\partial'^\mu\varphi'(x') - \frac{1}{2}m^2\varphi'^2(x)\right] \\
&= \int d^4x\, |J| \left[\frac{1}{2}\partial'_\mu\varphi(x)\partial'^\mu\varphi(x) - \frac{1}{2}m^2\varphi^2(x)\right] \\
&= \int d^4x\, \left(1-\partial_\rho a^\rho\right)\left[\frac{1}{2}\left(\partial_\mu\varphi(x)+\partial_\mu a^\nu\,\partial_\nu\varphi(x)\right)\left(\partial^\mu\varphi(x)+\partial^\mu a^\nu\,\partial_\nu\varphi(x)\right) - \frac{1}{2}m^2\varphi^2(x)\right] \\
&= \int d^4x\, \left[\mathcal{L}\left(\varphi(x),\partial_\mu\varphi(x)\right) - \partial_\rho a^\rho\,\mathcal{L}\left(\varphi(x),\partial_\mu\varphi(x)\right) + \partial_\mu a^\nu\,\partial_\nu\varphi(x)\partial^\mu\varphi(x) + O(a^2)\right] \\
&= I\left[\varphi(x)\right] + \int d^4x\, \left[-\partial_\rho a^\rho\,\mathcal{L}\left(\varphi(x),\partial_\mu\varphi(x)\right) + \partial_\mu a^\nu\,\partial_\nu\varphi(x)\partial^\mu\varphi(x)\right] + O(a^2),
\end{aligned}$$

This implies, to first order in a

$$\begin{aligned}
\delta I \equiv I\left[\varphi'(x')\right] - I\left[\varphi(x)\right] &= \int d^4x\, \left[-\partial_\rho a^\rho\,\mathcal{L}\left(\varphi(x),\partial_\mu\varphi(x)\right) + \partial_\mu a^\nu\,\partial_\nu\varphi(x)\partial^\mu\varphi(x)\right] \\
&= \int d^4x\, \partial_\mu a_\nu \left[\partial^\mu\varphi(x)\partial^\nu\varphi(x) - \eta^{\mu\nu}\mathcal{L}\left(\varphi(x),\partial_\mu\varphi(x)\right)\right] \\
&= \int d^4x\, \partial_\mu a_\nu\, \Theta^{\mu\nu},
\end{aligned}$$

where we have identified,

$$\Theta^{\mu\nu} \equiv \partial^\mu\varphi(x)\partial^\nu\varphi(x) - \eta^{\mu\nu}\mathcal{L}.$$

which is the Noether current, namely, the *canonical* stress-energy-momentum tensor for a real scalar field.

We note couple of things about this canonical stress tensor,

- One can go ahead and check, using the equations of motion, that this is indeed conserved that it satisfies the continuity equation, $\partial_\mu\theta^{\mu\nu} = 0$,

$$\begin{aligned}
\partial_\mu\Theta^{\mu\nu} &= \partial_\mu\left(\partial^\mu\varphi\right)\left(\partial^\nu\varphi\right) - \eta^{\mu\nu}\partial_\mu\mathcal{L} \\
&= \left(\partial^2\varphi\right)\left(\partial^\nu\varphi\right) + \left(\partial^\mu\varphi\right)\left(\partial_\mu\partial^\nu\varphi\right) - \partial^\nu\mathcal{L}
\end{aligned}$$

  We simplify the first term using the equations of motion,

$$\begin{aligned}
\partial^2\phi &= -\frac{\partial V}{\partial\varphi}, \\
\Rightarrow \left(\partial^2\varphi\right)\left(\partial^\nu\varphi\right) &= -\frac{\partial V}{\partial\varphi}\partial^\nu\varphi \\
&= -\partial^\nu V
\end{aligned}$$

  while the second term can be rewritten as

$$\left(\partial^\mu\varphi\right)\left(\partial_\mu\partial^\nu\varphi\right) = \left(\partial^\mu\varphi\right)\partial^\nu\left(\partial_\mu\varphi\right) = \partial^\nu\left(\frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi\right)$$

Thus, the first two terms add up to,

$$\begin{aligned}\left(\partial^2\varphi\right)\left(\partial^\nu\varphi\right)+\left(\partial^\mu\varphi\right)\left(\partial_\mu\partial^\nu\varphi\right) &= -\partial^\nu V+\partial^\nu\left(\frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi\right)\\ &= \partial^\nu\left(\frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi-V\right)\\ &= \partial^\nu\mathcal{L},\end{aligned}$$

which cancels the third term, giving us,

$$\partial_\mu\Theta^{\mu\nu}=0.$$

- The conserved charges corresponding to the current are nothing but the components of the four-momentum, P^μ, i.e.

$$P^\nu=\int d^3\mathbf{x}\,\Theta^{0\,\nu}.$$

Let's evaluate the 00-component i.e. *energy density*, θ^{00},

$$\begin{aligned}\Theta^{00} &= \frac{\partial\mathcal{L}}{\partial\left(\partial_0\varphi(x)\right)}\partial^0\varphi(x)-\eta^{00}\mathcal{L}\\ &= \left(\partial^0\varphi\right)^2-\mathcal{L},\\ &= \left(\partial^0\varphi(x)\right)^2-\frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi+V(\varphi),\\ &= \left(\partial_0\varphi\right)^2-\frac{1}{2}\left(\partial_0\varphi\right)^2+\frac{1}{2}\left(\boldsymbol{\nabla}\varphi\right)^2+V(\varphi),\\ &= \frac{1}{2}\left(\partial_0\varphi\right)^2+\frac{1}{2}\left(\boldsymbol{\nabla}\varphi\right)^2+V(\varphi).\end{aligned}$$

This expression being a sum of squares is manifestly positive. This is reassuring because we would want a free system to have energy positive semi-definite.

- For the real scalar field, the stress tensor is symmetric between the indices, μ,ν. This is only true for scalar fields and not true in general. For the Maxwell field, we will see that the corresponding stress tensor will *not* be symmetric.

- Note that the stress-energy-momentum is non-unique to some extent. One can always add a term like, $\partial_\lambda B^{\lambda\mu\nu}$ where B is a tensor that has the following antisymmetric properties,

$$B^{\lambda\mu\nu}=-B^{\mu\lambda\nu}.$$

The new quantity[1],

$$T^{\mu\nu}=\theta^{\mu\nu}+\partial_\lambda B^{\lambda\mu\nu}$$

[1]For those who are interested there is a special "symmetrizing improvement term", $B^{\lambda\mu\nu}$ is called the Belinfante-Rosenfield term after the two people who independently arrived at the expression. For the Maxwell field we will see that the variation of action under Lorentz transformation automatically gives us the improved symmetric stress tensor.

is also conserved,

$$\partial_\mu T^{\mu\nu} = \partial_\mu \theta^{\mu\nu} + \partial_\mu \partial_\lambda B^{\lambda\mu\nu} = 0.$$

For the Maxwell field, one can exploit this ambiguity to define a stress tensor which is symmetric in the indices, μ and ν,

$$T^{\mu\nu} = T^{\nu\mu}.$$

Before we do that we need to first obtain the expression for the charges conserved as a result of Lorentz invariance.

Homework 1: Follow the Noether algorithm to construct the conserved charges for the translation symmetry for the field theory of a generic tensor field, say $\mathcal{F}^{\cdots}_{\cdots}$ where the dots represent Lorentz or other indices, described by an action,

$$I\left[T^{\cdots}_{\cdots}\right] = \int d^4x \, \mathcal{L}\left(\mathcal{F}^{\cdots}_{\cdots}, \partial_\mu \mathcal{F}^{\cdots}_{\cdots}\right),$$

where the lagrangian density is a function of the field and its first order derivatives. [Hint: The answer should be

$$\Theta^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \mathcal{F}^{\cdots}_{\cdots}\right)} . \partial^\nu \mathcal{F}^{\cdots}_{\cdots} - \eta^{\mu\nu} \, \mathcal{L} \tag{4}$$

]

Homework 2: What is the expression for the canonical stress tensor for the Maxwell field, $A_\alpha(x)$. The lagrangian density for the Maxwell theory is,

$$\mathcal{L} = -\frac{1}{4} F_{\alpha\beta} \, F^{\alpha\beta}, \qquad F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha.$$

Homework 3: Follow the Noether algorithm to construct the conserved charges for the real scalar field theory for symmetry under Lorentz transformations,

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu \, x^\nu.$$

[Hint: The Noether algorithm should give,

$$\delta I = \int d^4x \, \partial_\mu \omega_{\alpha\beta} \, M^{\mu\alpha\beta}.$$

where $\omega_{\alpha\beta}$ is the infinitesimal parameter for Lorentz transformation,

$$\Lambda^\mu{}_\nu = \delta^\mu_\nu + \omega^\mu{}_\nu.$$

Thus the current $M^{\mu\alpha\beta}$ is the Noether current, a rank (3,0) tensor.]

Homework 4: Use the Noether algorithm to construct the Lorentz charges for the field theory of a generic field, $\mathcal{F}$, not necessarily a scalar.