Spring 2022: Quantum Field Theory (PH6418/ EP4618)

Notes for lecture 13*

April 11, 2022

1 Belinfante-Rosenfield symmetric stress tensor

Recall that using the Noether algorithm for spacetime translations, one can arrive at the expression for the canonical stress tensor (HW problem),

$$\Theta^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \mathcal{F}\right)}.\partial^\nu \mathcal{F} - \eta^{\mu\nu}\mathcal{L} \tag{1}$$

Here $\mathcal{F}$ is any type of field - could be a scalar, or a vector or a generic tensor field with various kinds of indices, Lorentz and internal. To reduce clutter of notation, the Lorentz indices or other indices of $\mathcal{F}$ are not explicitly displayed but they are understood to be there. In particular the dot "." between the tensors, $\frac{\partial\mathcal{L}}{\partial(\partial_\mu\mathcal{F})}$ and $\partial^\nu\mathcal{F}$ in the formula above indicates that all the indices of $\mathcal{F}$ in $\frac{\partial\mathcal{L}}{\partial(\partial_\mu\mathcal{F})}$ and being contracted with the indices of $\mathcal{F}$ in $\partial^\nu\mathcal{F}$:

$$\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \mathcal{F}\right)}.\partial^\nu \mathcal{F} = \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \mathcal{F}^{LMN\dots}\right)}\partial^\nu \mathcal{F}^{LMN\dots}.$$

For example, if F is a vector field (Maxwell field), say A_α, then,

$$\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \mathcal{F}\right)} \to \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\alpha)},$$

$$\partial^\nu\mathcal{F} \to \partial^\nu A_\alpha$$

and so the stress tensor for the vector field should be

$$\Theta^{\mu\nu}_A = \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \mathcal{F}\right)}.\partial^\nu \mathcal{F} - \eta^{\mu\nu}\mathcal{L} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\alpha)}\,\partial^\nu A_\alpha - \eta^{\mu\nu}\mathcal{L}.$$

By construction, the canonical stress tensor expression (1) is in general not symmetric under $\mu \to \nu$, i.e.

$$\Theta^{\mu\nu} \neq \Theta^{\nu\mu}.$$

One can check it for the Maxwell field for which,

$$\mathcal{L} = -\frac{1}{4}F_{\rho\sigma}F^{\rho\sigma}.$$

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Using this lagrangian density the canonical stress tensor expression for the Maxwell theory works out to be,

$$\Theta_A^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\alpha)}\,\partial^\nu A_\alpha - \eta^{\mu\nu}\mathcal{L} = -F^{\mu\alpha}\partial^\nu A_\alpha - \eta^{\mu\nu}\left(-\frac{1}{4}F_{\rho\sigma}F^{\rho\sigma}\right).$$

One can check that the energy density,

$$\begin{aligned}\Theta_A^{00} &= -F^{0j}\partial^0 A_j - \left(\frac{1}{4}F_{\rho\sigma}F^{\rho\sigma}\right)\\ &= E^j\left(E^j + \partial^j A^0\right) - \frac{1}{2}\left(\boldsymbol{E}^2 - \boldsymbol{B}^2\right)\\ &= \frac{1}{2}\left(\boldsymbol{E}^2 + \boldsymbol{B}^2\right) + \nabla\cdot(\boldsymbol{E}\Phi)\,.\end{aligned}$$

So up to a divergence it is a positive semidefinite quantity.

For various reasons, both physical and aesthetic, one would like to construct a symmetric stress tensor (e.g. the gravitational field couples to the symmetric stress tensor). A special symmetric stress tensor for matter fields was constructed by Belinfante (and later derived differently by Rosenfield) which couples to the gravitational field, the eponymous Belinfante-Rosenfield symmetric stress tensor. Here we review Belinfante's construction. To this end, recall that the stress-energy-momentum tensor, which is the Noether current for translations is ***non-unique***. One can construct a new stress-energy-momentum tensor from the canonical stress tensor, $\Theta^{\mu\nu}$ as follows,

$$T^{\mu\nu} = \Theta^{\mu\nu} + \partial_\lambda K^{\lambda\mu\nu}, \tag{2}$$

where $K^{\lambda\mu\nu}$ is an ***arbitrary*** rank 3 tensor which is antisymmetric the first pair of indices,

$$K^{\lambda\mu\nu} = -K^{\mu\lambda\nu}.$$

One can check right away that $T^{\mu\nu}$ too obeys the continuity equation:

$$\begin{aligned}\partial_\mu T^{\mu\nu} &= \partial_\mu\theta^{\mu\nu} + \underbrace{\partial_\mu\partial_\lambda K^{\lambda\mu\nu}}_{=0}\\ &= \partial_\mu\theta^{\mu\nu}\\ &= 0.\end{aligned}$$

One can also check that both stress-tensors lead to the identical conserved charge, namely the energy-momentum 4 vector:

$$P^\nu = \int d^3\boldsymbol{x}\;\theta^{0\nu} = \int d^3\boldsymbol{x}\;T^{0\nu}.$$

Our aim here is to exploit this ambiguity in the definition of the Noether current (stress tensor) to construct a new stress tensor which is symmetric, i.e.,

$$T^{\mu\nu} = T^{\nu\mu},$$

starting from the asymmetric canonical stress tensor, $\Theta^{\mu\nu}$. In particular we will follow Belinfante's prescription (1940) to arrive at a very special symmetrical stress tensor which is based on Lorentz symmetry and Noether currents for the Lorentz symmetry for a generic field theory.

The starting point is the Noether current for Lorentz transformations, namely the angular energy-momentum density tensor,

$$M^{\lambda\mu\nu} = x^{\mu}\Theta^{\lambda\nu} - x^{\nu}\Theta^{\lambda\mu} - i\,\frac{\partial\mathcal{L}}{\partial\left(\partial_{\lambda}\mathcal{F}\right)}.\Sigma^{\mu\nu}.\,\mathcal{F}, \tag{3}$$

where $\sum^{\mu\nu}$ is the spin matrix for the field, $\mathcal{F}$. Again the appearance of the "."'s imply a contraction of indices. For example, for the vector field, A_{α}

$$\frac{\partial\mathcal{L}}{\partial\left(\partial_{\lambda}F\right)}.\Sigma^{\mu\nu}.F = \frac{\partial\mathcal{L}}{\partial\left(\partial_{\lambda}A_{\alpha}\right)}\left(\Sigma^{\mu\nu}\right)_{\alpha}{}^{\beta}\,A_{\beta}.$$

The spin matrix can be easily found out from the infinitesimal form (linearized) of the Lorentz transformation law of the field $\mathcal{F}(x)$ under Lorentz transformations,

$$\begin{aligned} x^{\mu} \to x'^{\mu} &= x^{\mu} + \omega^{\mu}{}_{\nu}x^{\nu}, \\ \mathcal{F}(x) \to \mathcal{F}'(x') &= D(\Lambda).\mathcal{F}(x) \\ &= \mathcal{F}(x) - \frac{i}{2}\omega_{\mu\nu}\,\Sigma^{\mu\nu}.\mathcal{F}(x). \end{aligned}$$

Here we have used the infinitesimal version of the representation (matrix) $D(\Lambda)$

$$D(\Lambda) = I - \frac{i}{2}\omega_{\mu\nu}\,\Sigma^{\mu\nu}. \tag{4}$$

Since $\omega_{\mu\nu}$ is antisymmetric, then so must be the spin matrix,

$$\Sigma^{\mu\nu} = -\Sigma^{\nu\mu}.$$

The "."' imply a contraction of indices of $\left(\Sigma^{\mu\nu}\right)^{\cdots}$ and $\mathcal{F}_{\cdots}$. Again going back to our pet example, the vector field, A_{α},

$$A'_{\alpha}(x') = \Lambda_{\alpha}{}^{\beta}\,A_{\beta}(x) = A_{\alpha}(x) - \frac{i}{2}\omega_{\mu\nu}\left(\Sigma^{\mu\nu}\right)_{\alpha}{}^{\beta}\,A_{\beta}(x),$$

where the spin matrix turns out to be

$$\left(\Sigma^{\mu\nu}\right)_{\alpha}{}^{\beta} = i\left(\delta^{\mu}_{\alpha}\eta^{\nu\beta} - \eta^{\mu\beta}\delta^{\nu}_{\mu}\right).$$

From the expression (3) it is evident that this current is antisymmetric under a swap of the last pair of indices,

$$M^{\lambda\mu\nu} = -M^{\lambda\nu\mu}.$$

From the conservation law, $\partial_{\lambda}M^{\lambda\mu\nu} = 0$, one gets the antisymmetric part of the canonical stress tensor,

$$\Theta^{\mu\nu} - \Theta^{\nu\mu} = \partial_{\lambda}\left[i\,\frac{\partial\mathcal{L}}{\partial\left(\partial_{\lambda}\mathcal{F}\right)}.\Sigma^{\mu\nu}.\mathcal{F}\right], \tag{5}$$

a result we will use momentarily.

Our aim is to find a suitable $K^{\lambda\mu\nu}$ in (2) which will turn $T^{\mu\nu}$ symmetric, i.e.

$$T^{\mu\nu} = T^{\nu\mu},$$

which means that (in addition to K being addition to antisymmetric in the first two indices) we must have the following identity for K satisfied,

$$\Theta^{\mu\nu} + \partial_\lambda K^{\lambda\mu\nu} = \Theta^{\nu\mu} + \partial_\lambda K^{\lambda\nu\mu},$$

or, equivalently K must satisfy

$$\Theta^{\mu\nu} - \Theta^{\nu\mu} = -\partial_\lambda \left(K^{\lambda\mu\nu} - K^{\lambda\nu\mu}\right). \tag{6}$$

Comparing this condition for K with the result (5), we need K to satisfy the first order PDE,

$$\partial_\lambda \left(K^{\lambda\mu\nu} - K^{\lambda\nu\mu}\right) = -\partial_\lambda \left[i\frac{\partial \mathcal{L}}{\partial\left(\partial_\lambda \mathcal{F}\right)}.\Sigma^{\mu\nu}.\mathcal{F}\right].$$

Now a simple solution to this equation is,

$$K^{\lambda\mu\nu} - K^{\lambda\nu\mu} = -i\frac{\partial \mathcal{L}}{\partial\left(\partial_\lambda \mathcal{F}\right)}.\Sigma^{\mu\nu}.\mathcal{F}. \tag{7}$$

Permuting the indices λ, μ, ν cyclically once, we get,

$$K^{\mu\nu\lambda} - K^{\mu\lambda\nu} = -i\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \mathcal{F}\right)}.\Sigma^{\nu\lambda}.\mathcal{F}, \tag{8}$$

and permuting indices of this second equation cyclically once, we get,

$$K^{\nu\lambda\mu} - K^{\nu\mu\lambda} = -i\frac{\partial \mathcal{L}}{\partial\left(\partial_\nu \mathcal{F}\right)}.\Sigma^{\lambda\mu}.\mathcal{F}. \tag{9}$$

Adding both sides of equations (7) and (8) and then subtracting equation (9) from the sum, we get,

$$K^{\lambda\mu\nu} - K^{\lambda\nu\mu} + K^{\mu\nu\lambda} - \underbrace{K^{\mu\lambda\nu}}_{=-K^{\lambda\mu\nu}} - \left(\underbrace{K^{\nu\lambda\mu}}_{=-K^{\lambda\nu\mu}} - \underbrace{K^{\nu\mu\lambda}}_{=-K^{\mu\nu\lambda}}\right) = -i\frac{\partial \mathcal{L}}{\partial\left(\partial_\lambda \mathcal{F}\right)}.\Sigma^{\mu\nu}.\mathcal{F} - i\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \mathcal{F}\right)}.\underbrace{\Sigma^{\nu\lambda}}_{=-\Sigma^{\lambda\nu}}.\mathcal{F} + i\frac{\partial \mathcal{L}}{\partial\left(\partial_\nu \mathcal{F}\right)}.\Sigma^{\lambda\mu}.\mathcal{F}$$

or,

$$K^{\lambda\mu\nu} = \frac{i}{2}\left(\frac{\partial \mathcal{L}}{\partial\left(\partial_\nu \mathcal{F}\right)}.\Sigma^{\lambda\mu}.\mathcal{F} + \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \mathcal{F}\right)}.\Sigma^{\lambda\nu}.\mathcal{F} - \frac{\partial \mathcal{L}}{\partial\left(\partial_\lambda \mathcal{F}\right)}.\Sigma^{\mu\nu}.\mathcal{F}\right). \tag{10}$$

Thus, following Belinfante we have figured out the requisite antisymmetric tensor, K which is need to construct the symmetrical stress tensor,

$$T^{\mu\nu} = \Theta^{\mu\nu} + \partial_\lambda K^{\lambda\mu\nu}.$$

1.1 Rosenfield's prescription

Rosenfield independently, in the same year 1940, derived the same symmetrical stress tensor as Belinfante except using a different prescription - that of minimal coupling to gravity. He coupled the field theory to curved space metric using the minimal coupling principle, i.e. replacing all flat/Minkowski metric by a curved metric,

$$\eta_{\mu\nu} \to g_{\mu\nu}(x),$$

replacing the four-dimensional Lorentz invariant volume element by the general covariant volume element,

$$d^4x \to \sqrt{-g}d^4x,$$

and the partial derivatives by general covariant derivatives,

$$\partial_\mu \to \nabla_\mu.$$

Then he extracted the energy-momentum tensor by the usual general relativity formula,

$$T^{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta I}{\delta g_{\mu\nu}}.$$

and in the final expression restore flat space limit, i.e. $g_{\mu\nu} \to \eta_{\mu\nu}$.

Homework: Find out the symmetrical stress tensor for the Maxwell field (say $T_A^{\mu\nu}$) using Belinfante's prescription and also using Rosenfield's prescription. Show that it is conserved using the equation of motion (Maxwell equation in vacuo). What do the Lorentz symmetry Noether currents, $M^{\lambda\mu\nu}$ (3) look like when expressed in terms of the symmetrical stress tensor.

2 Local (gauge) symmetries

Often times, a field theory is invariant under continuous (differentiable) symmetry transformations where the symmetry parameter is a function of spacetime. In such cases we say that the action has a *local or gauge symmetry*. However, from the modern perspective such invariances of the action are not considered a genuine "symmetry" in the sense that the Hilbert space does not furnish a representation of the gauge symmetry and there is no associated Noether charge. An example of a field theory having a gauge symmetry is Maxwell theory,

$$I\left[A_\rho(x)\right] = \int d^4x \left(-\frac{1}{4}F_{\rho\sigma}F^{\rho\sigma}\right), \; F_{\rho\sigma} = \partial_\rho A_\sigma - \partial_\sigma A_\rho,$$

which is invariant under the local or gauge transformations of the vector potential,

$$A_\rho(x) \to A'_\rho(x) = A_\rho(x) - \partial_\rho \lambda(x) \tag{11}$$

where the symmetry transformation parameter, $\lambda(x)$, is a completely arbitrary scalar field (function of spacetime). If we apply the Noether algorithm to extract charges in this case, at the end of step 3 we arrive at the

$$0 = \delta I = \int d^4x \; (\partial_\mu \lambda) \; J^\mu.$$

$\delta I = 0$ holds identically because the action was symmetric (invariant) under local (gauge) transformations. So then we are forced to conclude that the Noether current and hence the Noether charge for the Maxwell field vanishes!

$$J^\mu = 0!$$

This vanishing current for gauge symmetry of Maxwell theory is responsible for the fact the photon (quantum of the Maxwell field) is neutral.

2.1 When can a local symmetry lead to a conserved current or charges?

If in the global limit of the gauge symmetry, i.e. when the symmetry parameter becomes a constant instead of a arbitrary function of spacetime, the fields have nontrivial transformation, then we have a global symmetry and the theory will admit nonzero Noether charges. An example of this is provided by the gluon field theory (nonabelian gauge theory). Here the gauge symmetry transformations are of the form,

$$A^i_\mu(x) \to A'^i_\mu(x) = A^i_\mu(x) + i f^i{}_{jk}\, \lambda^j(x)\, A^k_\mu(x) - \partial_\mu \lambda^i(x).$$

Here the $\lambda^i(x)$'s are the local symmetry transformation parameters, and $f^i{}_{jk}$ are structure constants of the nonabelian gauge group. For the abelian theory, $f^i{}_{jk} = 0$. The i, j, k indices are the color $SU(3)$indices. Clearly in the global limit, i.e. $\lambda^i(x) \to$ constant, the gluon fields, A^i_μ's transform nontrivially due to the non-derivative structure constant piece

$$A^i_\mu(x) \to A'^i_\mu(x) = A^i_\mu(x) + i f^i{}_{jk}\, \lambda^j\, A^k_\mu(x)).$$

Thus we have in this case a global symmetry and hence this theory will admit a conserved Noether charge, namely the *color charge*. This fact is responsible for gluons having a nontrivial color charge.