# PH6418/EP4618: Quantum Field Theory (Spring 2022) Midterm Exam*

May 10, 2022

1. **A.** Follow the Noether algorithm to construct the conserved charges for the translation symmetry for the field theory of a generic tensor field, say $\mathcal{F}(x)$, described by an action,

$$I\left[\mathcal{F}(x)\right] = \int d^4x \; \mathcal{L}\left(\mathcal{F}(x), \partial_\mu \mathcal{F}(x)\right), \tag{1}$$

where the lagrangian density is a function of the field and its first order derivatives. To reduce clutter of notation, the Lorentz or other internal indices of the field $\mathcal{F}$ are not being displayed. [Hint: The answer should be

$$\Theta^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \mathcal{F}\right)}.\partial^\nu \mathcal{F} - \eta^{\mu\nu}\,\mathcal{L} \tag{2}$$

where the dot "." denotes contraction over Lorentz or any other indices of $\mathcal{F}$]

**B.** Use the equation of motion of $\mathcal{F}$ to show that the canonical stress tensor is conserved,

$$\partial_\mu \Theta^{\mu\nu} = 0.$$

**C.** Use the general formula (2) to write down the expression for the canonical stress tensor of the Maxwell field, $A_\mu(x)$. Maxwell theory is given by the action,

$$I\left[A_\mu(x)\right] = \int d^4x \; \left(-\frac{1}{4} F_{\alpha\beta}F^{\alpha\beta}\right), \tag{3}$$

where, $F_{\alpha\beta}(x) = \partial_\alpha A_\beta(x) - \partial_\beta A_\alpha(x)$, is the Maxwell field strength tensor

**D.** Write down the expression for the linear momentum 4-vector for the Maxwell theory.
$(5 + 2 + 2 + 1 = 10$ points)

**Solution:** Under infinitesimal local translations,

$$x \to x' = x - \epsilon(x)$$

$$\mathcal{F}(x) \to \mathcal{F}'(x') = \mathcal{F}(x),$$

*(Maximum score: 50)

The derivative of the field transforms like,

$$\begin{aligned}\partial_\mu \mathcal{F}(x) \to \partial'_\mu \mathcal{F}'(x') &= \frac{\partial x^\nu}{\partial x'^\mu}\,\partial_\nu \mathcal{F}(x)\\ &= \left(\delta^\nu_\mu + \partial_\mu \epsilon^\nu\right)\partial_\nu \mathcal{F}(x)\\ &= \partial_\mu \mathcal{F}(x) + \partial_\mu \epsilon^\nu\,\partial_\nu \mathcal{F}(x).\end{aligned}$$

The Jacobian under local translations to first order,

$$\left|\frac{\partial x'^\mu}{\partial x^\nu}\right| = |\delta^\mu_\nu - \partial_\nu \epsilon^\mu| = 1 - \partial_\mu \epsilon^\mu.$$

Thus the first order change in the action under local translations is,

$$\begin{aligned}\delta I &= I\left[\mathcal{F}'(x')\right] - I\left[\mathcal{F}(x)\right]\\ &= \int d^4x'\,\mathcal{L}\left(\mathcal{F}'(x'), \partial'_\mu \mathcal{F}'(x')\right) - \int d^4x\,\mathcal{L}\left(\mathcal{F}(x), \partial_\mu \mathcal{F}(x)\right)\\ &= \int d^4x\,(1 - \partial_\mu \epsilon^\mu)\,\mathcal{L}\left(\mathcal{F}(x), \partial_\mu \mathcal{F}(x) + \partial_\mu \epsilon^\nu\,\partial_\nu \mathcal{F}(x)\right) - \int d^4x\,\mathcal{L}\left(\mathcal{F}(x), \partial_\mu \mathcal{F}(x)\right)\\ &= \int d^4x\,(1 - \partial_\mu \epsilon^\mu)\,\left[\mathcal{L}\left(\mathcal{F}(x), \partial_\mu \mathcal{F}(x)\right) + \partial_\mu \epsilon^\nu\,\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \mathcal{F}(x)\right)}.\partial_\nu \mathcal{F}(x) + O(\epsilon^2)\right] - \int d^4x\,\mathcal{L}\left(\mathcal{F}(x), \partial_\mu \mathcal{F}(x)\right)\\ &= \int d^4x\,\left[\partial_\mu \epsilon^\nu\,\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \mathcal{F}(x)\right)}.\partial_\nu \mathcal{F}(x) - \partial_\mu \epsilon^\mu\,\mathcal{L}\left(\mathcal{F}(x), \partial_\mu \mathcal{F}(x)\right)\right]\\ &= \int d^4x\,\partial_\mu \epsilon_\nu\,\left[\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \mathcal{F}(x)\right)}.\partial^\nu \mathcal{F}(x) - \eta^{\mu\nu}\,\mathcal{L}\left(\mathcal{F}(x), \partial_\mu \mathcal{F}(x)\right)\right].\end{aligned}$$

Thus the Noether current for translations, i.e. the canonical stress tensor is,

$$\Theta^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \mathcal{F}\right)}.\partial^\nu \mathcal{F} - \eta^{\mu\nu}\,\mathcal{L}.$$

**B.** The conservation law is proven using the EL equations for field $\mathcal{F}$ as follows

$$\begin{aligned}\partial_\mu \Theta^{\mu\nu} &= \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \mathcal{F}\right)}.\partial^\nu \mathcal{F} - \eta^{\mu\nu}\,\mathcal{L}\right)\\ &= \underbrace{\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \mathcal{F}\right)}\right)}_{=\frac{\partial \mathcal{L}}{\partial \mathcal{F}}}.\partial^\nu \mathcal{F} + \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \mathcal{F}\right)}.\partial^\mu \partial^\nu \mathcal{F} - \partial^\nu \mathcal{L}\\ &= \underbrace{\frac{\partial \mathcal{L}}{\partial \mathcal{F}}.\partial^\nu \mathcal{F} + \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \mathcal{F}\right)}.\partial^\mu \partial^\nu \mathcal{F}}_{\partial^\nu \mathcal{L}} - \partial^\nu \mathcal{L}\\ &= 0.\end{aligned}$$

**C.** For Maxwell theory, $\mathcal{L} = -\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}$ where $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$. Then the canonical stress tensor is,

$$\begin{aligned}\Theta^{\mu\nu} &= \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu A_\rho\right)}\partial^\nu A_\rho - \eta^{\mu\nu}\,\mathcal{L}\\ &= \frac{\partial}{\partial\left(\partial_\mu A_\rho\right)}\left(-\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}\right)\partial^\nu A_\rho - \eta^{\mu\nu}\,\mathcal{L}\\ &= -\frac{1}{2}\frac{\partial F_{\alpha\beta}}{\partial\left(\partial_\mu A_\rho\right)}F^{\alpha\beta}\partial^\nu A_\rho - \eta^{\mu\nu}\,\mathcal{L}\\ &= -\frac{1}{2}\left(\delta^\mu_\alpha \delta^\rho_\beta - \delta^\mu_\beta \delta^\rho_\alpha\right)F^{\alpha\beta}\partial^\nu A_\rho - \eta^{\mu\nu}\,\mathcal{L}\\ &= -F^{\mu\rho}\partial^\nu A_\rho - \eta^{\mu\nu}\,\mathcal{L}.\end{aligned}$$

**D.** Write down the expression for the linear momentum 4-vector for the Maxwell theory.

$$
\begin{aligned}
P^i = \int d^3\boldsymbol{x}\, \Theta^{0\,i} &= \int d^3\boldsymbol{x} \left( -F^{0\,\rho}\partial^i A_\rho - \cancelto{0}{\eta^{0\,i}} \mathcal{L} \right) \\
&= \int d^3\boldsymbol{x} \left( -F^{0\,j}\partial^i A_j \right) \\
&= \int d^3\boldsymbol{x} \left( \underbrace{-F^{0\,j}}_{E^i} \partial_i A^j \right) \\
&= \int d^3\boldsymbol{x}\, E^j \partial_i A^j \\
&= \int d^3\boldsymbol{x}\, E^j \left( \underbrace{\partial_i A^j - \partial_j A^i}_{\epsilon^{ijk} B^k} \right) + \int d^3\boldsymbol{x}\, E^j \partial_j A^i \\
&= \int d^3\boldsymbol{x}\, \epsilon^{ijk} E^j B^k + \cancelto{0}{\int d^3\boldsymbol{x}\, \partial_j \left(E^j A^i\right)} - \int d^3\boldsymbol{x}\, \cancelto{0}{\left(\partial_j E^j\right)} A^i \\
&= \int d^3\boldsymbol{x}\, \left(\boldsymbol{E} \times \boldsymbol{B}\right)^i .
\end{aligned}
$$

The second term is a total derivative and becomes a surface term at spatial infinity which vanishes due to boundary conditions, while the second term vanishes due to Gauss law in the absence of free charges, $\rho = 0 = \boldsymbol{\nabla} \cdot \boldsymbol{E}$.

2. Consider the theory of a Maxwell field, $A_\mu(x)$, coupled to an conserved external electric current-density (source), $j^\mu(x)$, given by the action,

$$
I\left[A_\mu(x)\right] = \int d^4x \left( -\frac{1}{4} F_{\alpha\beta}F^{\alpha\beta} - j^\mu A_\mu \right),
$$

**A.** Compute the dimension of the Maxwell field.

**B.** Use the functional form of the Euler-Lagrange equation, namely

$$
\frac{\delta L}{\delta \mathcal{F}(\boldsymbol{x})} = \frac{\partial}{\partial t} \left( \frac{\delta L}{\delta \dot{\mathcal{F}}(\boldsymbol{x})} \right)
$$

to arrive at the equation of motion for this theory. (Note that the rule for functional differentiation here would be

$$
\frac{\delta A_\mu(\boldsymbol{x})}{\delta A_\nu(\boldsymbol{y})} = \delta_\mu^\nu\, \delta^3\left(\boldsymbol{x} - \boldsymbol{y}\right)
$$

). Then use the alternative form of the Euler-Lagrange equation,

$$
\frac{\partial \mathcal{L}}{\partial \mathcal{F}} = \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \mathcal{F}\right)} \right)
$$

to arrive at the equation of motion (Maxwell equation). Here $\mathcal{F}$ is a generic tensor field with all indices suppressed to reduce clutter of notation.

**C.** Then work out the Hamiltonian for the Maxwell theory.

**D.** Show that the action is symmetric under the (abelian) gauge transformations,

$$A_\mu(x) \to A'_\mu(x) = A_\mu - \partial_\mu \lambda$$

where $\lambda(x)$ is an arbitrary scalar field.

(1 + 4 + 3 + 2 = 10 points)

**Solution: A.** The action is dimensionless. So

$$\left[d^4x\right] + \left[F_{\mu\nu}F^{\mu\nu}\right] = 0.$$

Now,

$$\left[d^4x\right] = \left[L^4\right] = -4,$$

while,

$$\left[F_{\mu\nu}F^{\mu\nu}\right] = 2\left[F_{\mu\nu}\right] = 2\left[\partial_\mu A_\nu\right] = 2\left[\frac{A_\mu}{L}\right] = 2\left[A_\mu\right] + 2.$$

So we have,

$$-4 + 2\left[A_\mu\right] + 2 = 0 \Rightarrow \left[A_\mu\right] = +1.$$

**B.** The LHS,

$$\begin{aligned}
\frac{\delta L}{\delta A_\mu(\boldsymbol{x})} &= \frac{\delta}{\delta A_\mu(\boldsymbol{x})}\left(-\frac{1}{4}\int d^3\boldsymbol{y}\, F_{\alpha\beta}(\boldsymbol{y})F^{\alpha\beta}(\boldsymbol{y}) - \int d^3\boldsymbol{y} j^\alpha(\boldsymbol{y})\, A_\alpha(\boldsymbol{y})\right) \\
&= -\frac{1}{4}\int d^3\boldsymbol{y}\frac{\delta\left(F_{\alpha\beta}(\boldsymbol{y})F^{\alpha\beta}(\boldsymbol{y})\right)}{\delta A_\mu(\boldsymbol{x})} - \int d^3\boldsymbol{y} j^\alpha(\boldsymbol{y}) \underbrace{\frac{\delta A_\alpha(\boldsymbol{y})}{\delta A_\mu(\boldsymbol{x})}}_{=\delta^\mu_\alpha \delta^3(\boldsymbol{x}-\boldsymbol{y})} \\
&= -\frac{1}{2}\int d^3\boldsymbol{y}\frac{\delta F_{\alpha\beta}(\boldsymbol{y})}{\delta A_\mu(\boldsymbol{x})}F^{\alpha\beta}(\boldsymbol{y}) - j^\mu(\boldsymbol{x}) \\
&= -\frac{1}{2}\int d^3\boldsymbol{y}\left[\frac{\delta\left(\partial_{y^\alpha}A_\beta(\boldsymbol{y})\right)}{\delta A_\mu(\boldsymbol{x})} - \frac{\delta\left(\partial_{y^\beta}A_\alpha(\boldsymbol{y})\right)}{\delta A_\mu(\boldsymbol{x})}\right]F^{\alpha\beta}(\boldsymbol{y}) - j^\mu(\boldsymbol{x}) \\
&= -\frac{1}{2}\int d^3\boldsymbol{y}\left[\partial_{y^i}\underbrace{\left(\frac{\delta A_\beta(\boldsymbol{y})}{\delta A_\mu(\boldsymbol{x})}\right)}_{=\delta^\mu_\beta \delta^3(\boldsymbol{y}-\boldsymbol{x})}F^{i\beta}(\boldsymbol{y}) - \partial_{y^j}\underbrace{\left(\frac{\delta A_\alpha(\boldsymbol{y})}{\delta A_\mu(\boldsymbol{x})}\right)}_{=\delta^\mu_\alpha \delta^3(\boldsymbol{y}-\boldsymbol{x})}F^{\alpha j}(\boldsymbol{y})\right] - j^\mu(\boldsymbol{x}) \\
&= \frac{1}{2}\int d^3\boldsymbol{y}\left[\delta^\mu_\beta\, \delta^3\left(\boldsymbol{y}-\boldsymbol{x}\right)\partial_{y^i}F^{i\beta}(\boldsymbol{y}) - \delta^\mu_\alpha\, \delta^3\left(\boldsymbol{y}-\boldsymbol{x}\right)\, \partial_{y^j}F^{\alpha j}(\boldsymbol{y})\right] - j^\mu(\boldsymbol{x}) \\
&= \frac{1}{2}\left(\partial_i F^{i\,\mu}(\boldsymbol{x}) - \partial_j F^{\mu\,j}(\boldsymbol{x})\right) - j^\mu(\boldsymbol{x}) \\
&= \partial_i F^{i\,\mu}(\boldsymbol{x}) - j^\mu(\boldsymbol{x}).
\end{aligned}$$

For the RHS we first need to compute,

$$
\begin{aligned}
\frac{\delta L}{\delta \dot{A}_\mu(\boldsymbol{x})} &= \frac{\delta}{\delta \dot{A}_\mu(\boldsymbol{x})} \left( -\frac{1}{4} \int d^3\boldsymbol{y}\, F_{\alpha\beta}(\boldsymbol{y}) F^{\alpha\beta}(\boldsymbol{y}) - \int d^3\boldsymbol{y} j^\alpha(\boldsymbol{y})\, A_\alpha(\boldsymbol{y}) \right) \\
&= -\frac{1}{4} \int d^3\boldsymbol{y} \frac{\delta \left( F_{\alpha\beta}(\boldsymbol{y}) F^{\alpha\beta}(\boldsymbol{y}) \right)}{\delta \dot{A}_\mu(\boldsymbol{x})} \\
&= -\frac{1}{2} \int d^3\boldsymbol{y} \frac{\delta F_{\alpha\beta}(\boldsymbol{y})}{\delta \dot{A}_\mu(\boldsymbol{x})}\, F^{\alpha\beta}(\boldsymbol{y}) \\
&= -\frac{1}{2} \int d^3\boldsymbol{y} \left[ \frac{\delta \left( \partial_{y^\alpha} A_\beta(\boldsymbol{y}) \right)}{\delta \dot{A}_\mu(\boldsymbol{x})} - \frac{\delta \left( \partial_{y^\beta} A_\alpha(\boldsymbol{y}) \right)}{\delta \dot{A}_\mu(\boldsymbol{x})} \right] F^{\alpha\beta}(\boldsymbol{y}) \\
&= -\frac{1}{2} \int d^3\boldsymbol{y} \left[ \underbrace{\frac{\delta \dot{A}_\beta(\boldsymbol{y})}{\delta \dot{A}_\mu(\boldsymbol{x})}}_{=\delta^\mu_\beta \delta^3(\boldsymbol{y}-\boldsymbol{x})} F^{0\,\beta}(\boldsymbol{y}) - \underbrace{\frac{\delta \dot{A}_\alpha(\boldsymbol{y})}{\delta \dot{A}_\mu(\boldsymbol{x})}}_{=\delta^\mu_\alpha \delta^3(\boldsymbol{y}-\boldsymbol{x})} F^{\alpha 0}(\boldsymbol{y}) \right] \\
&= -\frac{1}{2} \left( F^{0\,\mu}(\boldsymbol{x}) - F^{\mu\,0}(\boldsymbol{x}) \right) \\
&= F^{0\,\mu}(\boldsymbol{x}).
\end{aligned}
$$

Then the RHS,

$$
\partial_t \left( \frac{\delta L}{\delta \dot{A}_\mu(\boldsymbol{x})} \right) = -\partial_0 F^{0\,\mu}(\boldsymbol{x}).
$$

Thus the functional EL equations are,

$$
\partial_i F^{i\,\mu}(\boldsymbol{x}) - j^\mu(\boldsymbol{x}) = -\partial_0 F^{0\,\mu}(\boldsymbol{x}),
$$

or,

$$
\partial_0 F^{0\,\mu} + \partial_i F^{i\,\mu} = j^\mu
$$

or,

$$
\partial_\nu F^{\nu\mu} = j^\mu.
$$

Alternative EL equations without functional derivatives,

$$
\frac{\partial \mathcal{L}}{\partial \mathcal{F}} = \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial \left( \partial_\mu \mathcal{F} \right)} \right)
$$

The LHS:

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial A_\mu} &= \underbrace{\frac{\partial}{\partial A_\mu} \left( -\frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} \right)}_{\to 0} - \frac{\partial}{\partial A_\mu} \left( j^\alpha A_\alpha \right) \\
&= -j^\mu.
\end{aligned}
$$

For the RHS we need,

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \left( \partial_\nu A_\mu \right)} &= \frac{\partial}{\partial \left( \partial_\nu A_\mu \right)} \left( -\frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} \right) - \underbrace{\frac{\partial}{\partial \left( \partial_\nu A_\mu \right)} \left( j^\alpha A_\alpha \right)}_{\to 0} \\
&= -\frac{1}{2} \frac{\partial F_{\alpha\beta}}{\partial \left( \partial_\nu A_\mu \right)} F^{\alpha\beta} \\
&= -\frac{1}{2} \frac{\partial \left( \partial_\alpha A_\beta - \partial_\beta A_\alpha \right)}{\partial \left( \partial_\nu A_\mu \right)} F^{\alpha\beta} \\
&= -\frac{1}{2} \left( \delta^\nu_\alpha \delta^\mu_\beta - \delta^\nu_\beta \delta^\mu_\alpha \right) F^{\alpha\beta} \\
&= F^{\mu\nu}.
\end{aligned}
$$

Then the RHS,

$$\partial_\nu \left( \frac{\partial \mathcal{L}}{\partial \left( \partial_\nu A_\mu \right)} \right) = \partial_\nu F^{\mu\nu} = -\partial_\nu F^{\nu\,\mu}.$$

Thus the EL equations are,

$$-j^\mu = -\partial_\nu F^{\nu\,\mu},$$

or,

$$\partial_\nu F^{\nu\,\mu} = j^\mu.$$

**C.** Hamiltonian for the Maxwell theory: The Hamiltonian density in the absence of charges/currents is given by $\Theta^{00}$

$$\begin{aligned}
\Theta^{00} &= -F^{0\,\rho}\partial^0 A_\rho - \mathcal{L} \\
&= -F^{0\,i}\partial^0 A_i - \mathcal{L} \\
&= -F^{0\,i}\partial_0 A_i - \mathcal{L}
\end{aligned}$$

Now,

$$\mathcal{L} = -\frac{1}{4}F^2 = \frac{1}{2}\left(\boldsymbol{E}^2 - \boldsymbol{B}^2\right)$$

and,

$$F^{0\,i} = -E^i$$

while,

$$\begin{aligned}
\partial_0 A_i &= F_{0\,i} + \partial_i A^0 \\
&= E^i + \partial_i \Phi.
\end{aligned}$$

Substituting these in $\Theta^{00}$,

$$\begin{aligned}
\Theta^{00} &= E^i\left(E^i + \partial_i\Phi\right) - \frac{1}{2}\left(\boldsymbol{E}^2 - \boldsymbol{B}^2\right) \\
&= \frac{1}{2}\left(\boldsymbol{E}^2 + \boldsymbol{B}^2\right) + \boldsymbol{E}\cdot\boldsymbol{\nabla}\Phi \\
&= \frac{1}{2}\left(\boldsymbol{E}^2 + \boldsymbol{B}^2\right) + \boldsymbol{\nabla}\cdot\left(\boldsymbol{E}\Phi\right) - \Phi\underbrace{\boldsymbol{\nabla}\cdot\boldsymbol{E}}_{=0}
\end{aligned}$$

Thus the Hamiltonian is

$$\begin{aligned}
H &= \int d^3\boldsymbol{x}\,\Theta^{00} = \int d^3\boldsymbol{x}\,\frac{1}{2}\left(\boldsymbol{E}^2 + \boldsymbol{B}^2\right) + \cancelto{0}{\int d^3\boldsymbol{x}\,\boldsymbol{\nabla}\cdot\left(\boldsymbol{E}\Phi\right)} \\
&= \int d^3\boldsymbol{x}\,\frac{1}{2}\left(\boldsymbol{E}^2 + \boldsymbol{B}^2\right).
\end{aligned}$$

**D.** The Maxwell field strength $F_{\mu\nu}$ is invariant under gauge transformations:

$$A_\mu(x) \to A'_\mu(x) = A_\mu - \partial_\mu\lambda$$

where $\lambda(x)$ is an arbitrary scalar field. So the term $-\frac{1}{4}F^2$ in the lagrangian is invariant under gauge transformations. Now let's look at the coupling term, $j^\mu A_\mu$. Under a gauge transformation this transforms to,

$$\begin{aligned}
j^\mu A_\mu \to j^\mu\,A'_\mu &= j^\mu\,\left(A_\mu - \partial_\mu\lambda\right) \\
&= j^\mu A_\mu - j^\mu\partial_\mu\lambda \\
&= j^\mu A_\mu - \partial_\mu\left(j^\mu\,\lambda\right) + \lambda\cancelto{0}{\left(\partial_\mu j^\mu\right)}.
\end{aligned}$$

The last term vanished since $j^\mu$is a conserved current. So the coupling term changes a total derivative. When integrated against spacetime this would reduce to a surface term at infinity which will be made to vanish by appropriate initial and boundary conditions. Thus,

$$\int d^4x\, j^\mu\, A'_\mu = \int d^4x\, j^\mu\, A_\mu,$$

and hence the Maxwell equation with coupling to a conserved current is invariant under gauge transformations.

3. Apply the Noether algorithm to construct the conserved charges for the Lorentz invariant field theory (1) for a generic field (representation of Lorentz group) $\mathcal{F}(x)$, for symmetry under Lorentz transformations,

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu\, x^\nu,$$
$$\mathcal{F}(x) \to \mathcal{F}'(x') = D\left(\Lambda\right)\, \mathcal{F}(x).$$

where the representation (matrix) $D(\Lambda)$ is generated by the $\Sigma^{\mu\nu}$ is the spin matrix (generator):

$$D(\Lambda) = \exp\left(-\frac{i}{2}\,\omega_{\alpha\beta}\,\Sigma^{\alpha\beta}\right) \approx \mathbb{1} - \frac{i}{2}\,\omega_{\alpha\beta}\,\Sigma^{\alpha\beta}.$$

[Hint: The Noether algorithm should give,

$$\delta I = \int d^4x\, \frac{1}{2}\,\partial_\lambda \omega_{\mu\nu}\; M^{\lambda\,\mu\,\nu}.$$

where $\omega_{\mu\nu}$ is the infinitesimal parameter for Lorentz transformation,

$$\Lambda^\mu{}_\nu = \delta^\mu_\nu + \omega^\mu{}_\nu.$$

and,

$$M^{\lambda\,\mu\,\nu} = x^\mu\,\Theta^{\lambda\,\nu} - x^\nu\,\Theta^{\lambda\,\mu} - i\,\frac{\partial\mathcal{L}}{\partial(\partial_\lambda\mathcal{F})}.\Sigma^{\mu\nu}.\mathcal{F}.$$

Thus the current $M^{\lambda\,\mu\,\nu}$ is the Noether current, a rank (3,0) tensor.]

(10 points)

**Solution:** The infinitesimal form of the local Lorentz transformation is,

$$x^\mu \to x'^\mu = x^\mu + \omega^\mu{}_\nu(x)\, x^\nu,$$

$$\begin{aligned}\mathcal{F}(x) \to \mathcal{F}'(x') &= \left(\mathbb{1} - \frac{i}{2}\,\omega_{\alpha\beta}(x)\,\Sigma^{\alpha\beta}\right).\mathcal{F}(x)\\ &= \mathcal{F}(x) - \frac{i}{2}\,\omega_{\alpha\beta}(x)\,\Sigma^{\alpha\beta}.\mathcal{F}(x).\end{aligned}$$

The derivative of the field transforms to,

$$\begin{aligned}\partial_\mu\mathcal{F}(x) \to \partial'_\mu\mathcal{F}'(x') &= \frac{\partial x^\nu}{\partial x'^\mu}\,\partial_\nu\left(D\left(\Lambda\right)\,\mathcal{F}(x)\right)\\ &= \left(\delta^\nu_\mu - \omega^\nu{}_\mu - \partial_\mu\omega^\nu{}_\alpha x^\alpha\right)\left(\partial_\nu\mathcal{F} - \frac{i}{2}\,\partial_\nu\omega_{\alpha\beta}\,\Sigma^{\alpha\beta}.\mathcal{F} - \frac{i}{2}\,\omega_{\alpha\beta}(x)\,\Sigma^{\alpha\beta}.\partial_\nu\mathcal{F}\right)\\ &= \partial_\mu\mathcal{F}(x)\underbrace{-\partial_\mu\omega^\nu{}_\alpha x^\alpha\,\partial_\nu\mathcal{F}(x) - \frac{i}{2}\,\partial_\mu\omega_{\alpha\beta}\,\Sigma^{\alpha\beta}.\mathcal{F} - \omega^\nu{}_\mu\partial_\nu\mathcal{F}(x) - \frac{i}{2}\,\omega_{\alpha\beta}\,\Sigma^{\alpha\beta}.\partial_\mu\mathcal{F}}_{\equiv\Delta(\partial_\mu\mathcal{F}(x))}.\\ &= \partial_\mu\mathcal{F}(x) + \Delta\left(\partial_\mu\mathcal{F}(x)\right)\end{aligned}$$

Finally the Jacobian of the infinitesimal local Lorentz transformation,

$$\left|\frac{\partial x'^\mu}{\partial x^\nu}\right| = |\delta^\mu_\nu + \omega^\mu{}_\nu + \partial_\nu \omega^\mu{}_\alpha x^\alpha| = 1 + \cancelto{0}{\omega^\mu{}_\mu} + \partial_\mu \omega^\mu{}_\nu\, x^\nu = 1 + \partial_\mu \omega^\mu{}_\nu\, x^\nu.$$

The first order change in the action for an infinitesimal local Lorentz transformation is then,

$$\begin{aligned}
\delta I &= I\left[\mathcal{F}'(x')\right] - I\left[\mathcal{F}(x)\right] \\
&= \int d^4x'\, \mathcal{L}\left(\mathcal{F}'(x'), \partial'_\mu \mathcal{F}'(x')\right) - \int d^4x\, \mathcal{L}\left(\mathcal{F}(x), \partial_\mu \mathcal{F}(x)\right) \\
&= \int d^4x\, \left(1 + \partial_\mu \omega^\mu{}_\nu\, x^\nu\right) \mathcal{L}\left(\mathcal{F}(x) - \frac{i}{2}\,\omega_{\alpha\beta}\,\Sigma^{\alpha\beta}.\mathcal{F}(x), \partial_\mu \mathcal{F}(x) + \Delta\left(\partial_\mu \mathcal{F}(x)\right)\right) - \int d^4x\, \mathcal{L}\left(\mathcal{F}(x), \partial_\mu \mathcal{F}(x)\right) \\
&= \int d^4x\, \left[\partial_\mu \omega^\mu{}_\nu\, x^\nu \mathcal{L} - \frac{i}{2}\,\omega_{\alpha\beta}\, \frac{\partial \mathcal{L}}{\partial \mathcal{F}}.\Sigma^{\alpha\beta}.\mathcal{F} + \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \mathcal{F}\right)}.\Delta\left(\partial_\mu \mathcal{F}(x)\right)\right] \\
&= \int d^4x\, \partial_\mu \omega_{\nu\rho}\left(\eta^{\mu\nu} x^\rho \mathcal{L} - \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \mathcal{F}\right)}.\partial^\nu \mathcal{F}(x)\, x^\rho - \frac{i}{2} - \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \mathcal{F}\right)}.\Sigma^{\nu\rho}.\mathcal{F}\right) + \ldots \\
&= \int d^4x\, \partial_\mu \omega_{\nu\rho}\left(-x^\rho \Theta^{\mu\nu} - \frac{i}{2}\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \mathcal{F}\right)}.\Sigma^{\nu\rho}.\mathcal{F}\right) + \ldots \\
&= \int d^4x\, \frac{1}{2}\partial_\mu \omega_{\nu\rho}\left(x^\nu \Theta^{\mu\rho} - x^\rho \Theta^{\mu\nu} - i\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \mathcal{F}\right)}.\Sigma^{\nu\rho}.\mathcal{F}\right) + \ldots.
\end{aligned}$$

The "..." represent terms which are proportional to $\omega$ and not derivatives of $\omega$. These terms are guaranteed to vanish due to the existence of the global symmetry. Then evidently the conserved current for the Lorentz transformation is,

$$M^{\mu\nu\rho} = \underbrace{x^\nu \Theta^{\mu\rho} - x^\rho \Theta^{\mu\nu}}_{\text{Orbital}} \underbrace{-i\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \mathcal{F}\right)}.\Sigma^{\nu\rho}.\mathcal{F}}_{\text{Spin}}$$

The conserved charges for this are the angular momenta,

$$J^{\nu\rho} = \int d^3\boldsymbol{x}\, M^{0\,\nu\,\rho} = \int d^3\boldsymbol{x}\, \left(x^\nu \Theta^{0\rho} - x^\rho \Theta^{0\nu} - i\frac{\partial \mathcal{L}}{\partial \dot{\mathcal{F}}}.\Sigma^{\nu\rho}.\mathcal{F}\right).$$

4. A complex scalar field $\Phi(x)$ has the following Lagrange density,

$$\mathcal{L} = \left(\partial^\mu \Phi\right)^* \left(\partial_\mu \Phi\right) - m^2 \Phi^* \Phi - \lambda\, \left(\Phi^* \Phi\right)^2 - \mu^2\left(\Phi^2 + \Phi^{*\,2}\right), \qquad m^2 > 2\mu^2.$$

**A.** Write down the continuous global symmetry when $\mu^2 = 0$. Write down the corresponding Noether current $J^\mu$(no derivation necessary).

**B.** Obtain the 4-divergence of $J^\mu$ when $\mu^2 \neq 0$.

**C.** What is the physical interpretation of the free Lagrangian density (i.e $\lambda = 0$) when $\mu^2 \neq 0$.

(4 + 4 + 2 = 10 points)

**Solution:** **A.** The continuous global symmetry when $\mu^2 = 0$, is the global U(1) symmetry (symmetry under phase transformations):

$$\Phi \to \Phi' = e^{-i\alpha}\Phi.$$

The Noether current for this symmetry is,

$$J^\mu = i\left(\Phi^\dagger \partial^\mu \Phi - \partial^\mu \Phi^\dagger\, \Phi\right)$$

**B.** The equation of motion in the presence of the $\mu^2$ term is,

$$\frac{\partial \mathcal{L}}{\partial \Phi} = \partial_\mu \left[ \frac{\partial \mathcal{L}}{\partial \left( \partial_\mu \Phi \right)} \right]$$

or,

$$-m^2 \Phi^* - 2\lambda \left( \Phi^* \Phi \right) \Phi^* - 2\mu^2 \Phi = \Box \Phi^*$$

or,

$$\Box \Phi^* = -m^2 \Phi^* - 2\lambda \left| \Phi \right|^2 \Phi^* - 2\mu^2 \Phi .$$

Similarly for $\Phi$ we get,

$$\Box \Phi = -m^2 \Phi - 2\lambda \left| \Phi \right|^2 \Phi - 2\mu^2 \Phi^* .$$

Now we compute the 4-divergence of the current,

$$\begin{aligned} \partial_\mu J^\mu &= i \left( \Phi^* \, \Box \Phi - \Box \Phi^* \, \Phi \right) \\ &= -i \left( m^2 \left| \Phi \right|^2 + 2\lambda \left( \left| \Phi \right|^2 \right)^2 + 2\mu^2 \Phi^{*2} - m^2 \left| \Phi \right|^2 - 2\lambda \left( \left| \Phi \right|^2 \right)^2 - 2\mu^2 \Phi^2 \right) \\ &= i \, 2\mu^2 \left( \Phi^2 - \Phi^{*2} \right) . \end{aligned}$$

**C.** Resolving the complex scalar $\Phi$ into real and imaginary components,

$$\Phi = \frac{\varphi_1 + i \varphi_2}{\sqrt{2}}$$

and plugging into the lagrangian with the $\mu^2$ term turned on while having the $\lambda$-term turned off, we get the lagrangian to read as,

$$\mathcal{L} = \frac{1}{2} \left( \partial \varphi_1 \right)^2 - \frac{1}{2} \left( m^2 + 2\mu^2 \right) \varphi_1^2 + \frac{1}{2} \left( \partial \varphi_2 \right)^2 - \frac{1}{2} \left( m^2 - 2\mu^2 \right) \varphi_2^2 .$$

This is the lagrangian for two free real scalar fields $\varphi_1, \varphi_2$ of different masses, $m_1^2 = m^2 + 2\mu^2$ and $m_2^2 = m^2 - 2\mu^2$ respectively.

5. Consider the complex scalar field theory described by the action,

$$I \left[ \Phi(x) \right] = \int d^4 x \ \left[ \left( \partial^\mu \Phi \right)^* \partial_\mu \Phi - m^2 \Phi^* \Phi - V \left( \Phi^* \Phi \right) \right] . \tag{4}$$

which has the global $U(1)$ symmetry,

$$\Phi(x) \to \Phi'(x) = e^{-i \alpha} \, \Phi(x) . \tag{5}$$

**A.** Show that when the system is expressed in terms of the real and imaginary parts, the complex scalar field, $\varphi_1, \varphi_2$ as defined by

$$\Phi = \frac{\varphi_1 + i \, \varphi_2}{\sqrt{2}}$$

then the $U(1)$ symmetry transformation 5 looks like an $SO(2)$ transformation, namely,

$$\varphi \to \varphi\prime = O \, \varphi ,$$

where

$$\varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}$$

is a column vector and $O$ is a $2\times 2$ matrix orthogonal matrix of unit determinant (an element of $SO(2)$). This shows the isomorphism of the groups, $U(1)\cong SO(2)$.

**B.** Work out the Noether current(s) for this $SO(2)$ symmetry. You will first need to rewrite the action (4) in terms of the real scalar field column vector $\varphi$:

$$I\left[\varphi(x)\right]=\int d^4x\,\left[\frac{1}{2}\left(\partial^{\mu}\varphi\right)^T\left(\partial_{\mu}\varphi\right)-\frac{m^2}{2}\varphi^T\varphi-V\left(\varphi^T\varphi\right)\right] \tag{6}$$

where $\varphi^T=\text{transpose}(\varphi)$ is a row vector.

**C.** The equation of motion to show that the Noether current is conserved i.e. satisfy continuity equation.

**D.** The action (6) is actually symmetric under $O(2)$ transformations not just $SO(2)$. Since $O(2)=P\cup SO(2)$, where $P$ is the (field space) parity transformation,

$$P=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

What is this symmetry in terms of the complex field.

(3 + 3 + 3 + 1 = 10 points)

**Solution:** A. Resolving the symmetry transformed field, $\Phi'$ in terms of real and imaginary components, $\varphi_1',\varphi_2'$, we can write the symmetry transformation $\Phi'(x)=e^{-i\,\alpha}\,\Phi(x)$ as,

$$\begin{aligned}\left(\varphi_1'+i\,\varphi_2'\right)&=e^{-i\alpha}\,\left(\varphi_1+i\,\varphi_2\right)\\&=\left(\cos\alpha\,\varphi_1+\sin\alpha\,\varphi_2\right)+i\,\left(-\sin\alpha\,\varphi_1+\cos\alpha\,\varphi_2\right).\end{aligned}$$

So in terms of the components, the symmetry transformation reads,

$$\begin{aligned}\varphi_1\to\varphi_1'&=\cos\alpha\,\varphi_1+\sin\alpha\,\varphi_2,\\ \varphi_2\to\varphi_2'&=-\sin\alpha\,\varphi_1+\cos\alpha\,\varphi_2.\end{aligned}$$

In matrix notation,

$$\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}\to\begin{pmatrix}\varphi_1'\\ \varphi_2'\end{pmatrix}=\underbrace{\begin{pmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{pmatrix}}_{O}\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}.$$

Evidently the matrix $O$ is an element of $SO(2)$ since it is same as the rotation matrix in 2 dimensions (rotation by an angle $\alpha$ in the $xy$ plane).

**B.** Noether current for $SO(2)$ symmetry: The action is

$$I\left[\varphi(x)\right]=\int d^4x\,\left[\frac{1}{2}\left(\partial^{\mu}\varphi\right)^T\left(\partial_{\mu}\varphi\right)-\frac{m^2}{2}\varphi^T\varphi-V\left(\varphi^T\varphi\right)\right]$$

where now $\varphi=\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}$. We need to compute change in the action when the symmetry parameter $\alpha$ is made a function of spacetime, $\alpha(x)$ ,

$$\delta I=I\left[\varphi'\right]-I\left[\varphi\right],\varphi'=O(\alpha(x))\,\varphi$$

Clearly the potential terms, $-\frac{m^2}{2}\varphi^T\varphi-V\left(\varphi^T\varphi\right)$ are invariant even under local $SO(2)$ since they contain no derivatives of the field. So the only nonzero change in the action arises from the kinetic term,

$$\delta I=\int d^4x\,\left[\frac{1}{2}\left(\partial^{\mu}\varphi'\right)^T\left(\partial_{\mu}\varphi'\right)-\frac{1}{2}\left(\partial^{\mu}\varphi\right)^T\left(\partial_{\mu}\varphi\right)\right].$$

To leading order in $\alpha$,

$$\varphi' = \varphi + \alpha(x)\, A\, \varphi,\ A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Thus,

$$\partial_\mu \varphi' = \partial_\mu \varphi + \partial_\mu \alpha\, A\, \varphi + \alpha\, A\, \partial_\mu \varphi$$

and,

$$\begin{aligned}
\left(\partial^\mu \varphi'\right)^T \left(\partial_\mu \varphi'\right) &= \left(\partial^\mu \varphi + \partial^\mu \alpha\, A\, \varphi + \alpha\, A\, \partial^\mu \varphi\right)^T \left(\partial_\mu \varphi + \partial_\mu \alpha\, A\, \varphi + \alpha\, A\, \partial_\mu \varphi\right) \\
&= \left(\partial^\mu \varphi^T + \partial^\mu \alpha\, \varphi^T\, A^T + \alpha\, \partial^\mu \varphi^T\, A^T\right) \left(\partial_\mu \varphi + \partial_\mu \alpha\, A\, \varphi + \alpha\, A\, \partial_\mu \varphi\right) \\
&= \left(\partial^\mu \varphi^T - \partial^\mu \alpha\, \varphi^T\, A - \alpha\, \partial^\mu \varphi^T\, A\right) \left(\partial_\mu \varphi + \partial_\mu \alpha\, A\, \varphi + \alpha\, A\, \partial_\mu \varphi\right) \\
&= \left(\partial^\mu \varphi^T\right) \left(\partial_\mu \varphi\right) + \partial_\mu \alpha\, \left(\partial^\mu \varphi^T A\, \varphi - \varphi^T\, A\, \partial^\mu \varphi\right) + O\left(\alpha^2\right).
\end{aligned}$$

Substituting back in the action (change) we get,

$$\delta I = \int d^4x\, \partial_\mu \alpha\, \frac{1}{2} \left(\partial^\mu \varphi^T A\, \varphi - \varphi^T\, A\, \partial^\mu \varphi\right).$$

Evidently the Noether current is,

$$J^\mu = \frac{1}{2} \left(\partial^\mu \varphi^T A\, \varphi - \varphi^T\, A\, \partial^\mu \varphi\right) = \varphi_2 \partial^\mu \varphi_1 - \varphi_1 \partial^\mu \varphi_2.$$

**C.** Check that the $SO(2)$ Noether current is conserved i.e. satisfy continuity equation: For this we need the equations of motion for $\varphi_1, \varphi_2$. The Lagrangian density, in terms of $\varphi_1, \varphi_2$ is,

$$\mathcal{L} = \left(\partial^\mu \Phi\right)^* \partial_\mu \Phi - m^2 \Phi^* \Phi - V(\Phi^* \Phi) = \frac{1}{2} \left(\partial \varphi_1\right)^2 + \frac{1}{2} \left(\partial \varphi_2\right)^2 - \frac{m^2}{2} \left(\varphi_1^2 + \varphi_2^2\right) - V(\rho),\ \rho = \varphi_1^2 + \varphi_2^2.$$

Then the EL equation for $\varphi_1$ is,

$$\partial_\mu \left( \frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \varphi_1\right)} \right) = \frac{\partial \mathcal{L}}{\partial \varphi_1}$$

or,

$$\begin{aligned}
\Box \varphi_1 &= m^2 \varphi_1 - \frac{dV}{d\rho} \frac{\partial \rho}{\partial \varphi_1} \\
&= m^2 \varphi_1 - 2 \frac{dV}{d\rho}\, \varphi_1.
\end{aligned}$$

By symmetry the EL equation for $\varphi_2$ is then,

$$\Box \varphi_2 = m^2 \varphi_2 - 2 \frac{dV}{d\rho}\, \varphi_2.$$

Now we compute the 4-divergence of the current,

$$\begin{aligned}
\partial_\mu J^\mu &= \varphi_2 \Box \varphi_1 - \varphi_1 \Box \varphi_2 \\
&= \varphi_2 \left( m^2 \varphi_1 - 2 \frac{dV}{d\rho}\, \varphi_1 \right) - \varphi_1 \left( m^2 \varphi_2 - 2 \frac{dV}{d\rho}\, \varphi_2 \right) \\
&= 0.
\end{aligned}$$

**D.** The configuration space parity transformation,

$$\begin{aligned}
\varphi' &= P\, \varphi \\
\begin{pmatrix} \varphi_1' \\ \varphi_2' \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \\
&= \begin{pmatrix} \varphi_1 \\ -\varphi_2 \end{pmatrix}.
\end{aligned}$$

in terms of complex scalar, becomes,

$$\Phi \to \Phi' = \frac{\varphi_1' + i\,\varphi_2'}{\sqrt{2}} = \frac{\varphi_1 - i\,\varphi_2}{\sqrt{2}} = \Phi^\dagger .$$

This is the complex conjugation (charge conjugation) symmetry of the complex scalar field theory.