PH6418/ PH4618: Quantum Field Theory (Spring 2022)

Notes for Lecture 16: Correlation functions of the free quantum scalar field*

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1 Correlation functions in free scalar field theory

The result of any measurement involving the scalar field theory can be in terms of vacuum expectation values of a string of say $n$field operators at $n$different points in spacetime,

$$\langle 0 \big| \varphi(x_1)\, \varphi(x_2) \ldots \varphi(x_n) \big| 0 \rangle$$

for arbitrary n. These are also called n-point correlation functions or simply n-point functions. These are the fundamental quantities to be computed on the theory side to facilitate comparison with experiments. Let's compute some low order correlation functions in the free scalar field theory. Even before we compute anything, just on the basis of the (discrete, internal) sign reflection symmetry,

$$\varphi(x) \to -\varphi(x), \quad \forall x,$$

we can say that *odd order correlation functions vanish.*

$$\langle 0 \big| \varphi(x_1)\, \varphi(x_2) \ldots \varphi(x_n) \big| 0 \rangle = 0, \quad n = \text{odd}.$$

The proof is as follows. Under the sign reflection symmetry all physical observables including the correlation functions must remain unchanged. Under sign reflection symmetry,

$$\langle 0 \big| \varphi(x_1)\, \varphi(x_2) \ldots \varphi(x_n) \big| 0 \rangle \to (-)^n \, \langle 0 \big| \varphi(x_1)\, \varphi(x_2) \ldots \varphi(x_n) \big| 0 \rangle$$

For odd n,

$$\langle 0 \big| \varphi(x_1)\, \varphi(x_2) \ldots \varphi(x_n) \big| 0 \rangle = -\langle 0 \big| \varphi(x_1)\, \varphi(x_2) \ldots \varphi(x_n) \big| 0 \rangle,$$

which means,

$$\langle 0 \big| \varphi(x_1)\, \varphi(x_2) \ldots \varphi(x_n) \big| 0 \rangle = 0, \quad n = \text{odd}.$$

So the first nontrivial thing to compute are the various two point correlation functions,

$$\langle 0 \big| \varphi(x)\, \varphi(y) \big| 0 \rangle.$$

Since in quantum mechanics while taking products of two operators the ordering of operators is important, we will work out each case in some detail in the following sections.

*Typos and errors should be reported to sroy@phy.iith.ac.in

1.1 Wightman functions

The Wightman functions are defined to be,

$$\Delta_+(x,y) = \langle 0|\varphi(x)\varphi(y)|0\rangle,$$
$$\Delta_-(x,y) = \langle 0|\varphi(y)\varphi(x)|0\rangle.$$

To evaluate the two point correlation function $\langle 0|\varphi(x)\,\varphi(y)|0\rangle$, we will separately evaluate the ket, $\varphi(y)|0\rangle$ and the bra $\langle 0|\varphi(x)$ and then take the product,

$$\begin{aligned}\varphi(y)|0\rangle &= \int \frac{d^3\boldsymbol{p}}{\sqrt{(2\pi)^3\, 2\omega_{\boldsymbol{p}}}} \left(a(\boldsymbol{p})\; e^{-ip.y} + a^\dagger(\boldsymbol{p}) e^{ip.y}\right)_{p^0=\omega_{\boldsymbol{p}}} |0\rangle \\ &= \int \frac{d^3\boldsymbol{p}}{\sqrt{(2\pi)^3\, 2\omega_{\boldsymbol{p}}}}\; e^{ip.y}\Big|_{p^0=\omega_{\boldsymbol{p}}}\; a^\dagger(\boldsymbol{p})|0\rangle.\end{aligned}$$

Taking the hermitian conjugate,

$$\langle 0|\varphi(x) = \left(\varphi(x)|0\rangle\right)^\dagger = \int \frac{d^3\boldsymbol{k}}{\sqrt{(2\pi)^3\, 2\omega_{\boldsymbol{k}}}}\; e^{-ik.x}\Big|_{k^0=\omega_{\boldsymbol{k}}} \langle 0|a(\boldsymbol{k}).$$

Thus the Wightman function is

$$\begin{aligned}\Delta_+(x,y) \equiv \langle 0|\varphi(x)\varphi(y)|0\rangle &= \int \frac{d^3\boldsymbol{k}\; d^3\boldsymbol{p}}{(2\pi)^3\, 2\sqrt{\omega_{\boldsymbol{k}}\omega_{\boldsymbol{p}}}} \left(e^{-ik.x}\Big|_{k^0=\omega_{\boldsymbol{k}}}\; e^{ip.y}\Big|_{p^0=\omega_{\boldsymbol{p}}}\right) \langle 0|a(\boldsymbol{k})a^\dagger(\boldsymbol{p})|0\rangle \\ &= \int \frac{d^3\boldsymbol{k}\; d^3\boldsymbol{p}}{(2\pi)^3\, 2\sqrt{\omega_{\boldsymbol{k}}\omega_{\boldsymbol{p}}}} \left(e^{-ik.x}\Big|_{k^0=\omega_{\boldsymbol{k}}}\; e^{ip.y}\Big|_{p^0=\omega_{\boldsymbol{p}}}\right) \langle 0|\underbrace{\left[a(\boldsymbol{k}), a^\dagger(\boldsymbol{p})\right]}_{=\delta^3(\boldsymbol{k}-\boldsymbol{p})}|0\rangle \\ &= \int \frac{d^3\boldsymbol{k}}{(2\pi)^3\, 2\omega_{\boldsymbol{k}}}\; e^{-ik.(x-y)}\Big|_{k^0=\omega_{\boldsymbol{k}}}. \qquad (1)\end{aligned}$$

Then the reverse ordered Wightman function is,

$$\begin{aligned}\Delta_-(x,y) \equiv \langle 0|\varphi(y)\varphi(x)|0\rangle &= \Delta_+(y,x) \\ &= \int \frac{d^3\boldsymbol{k}}{(2\pi)^3\, 2\omega_{\boldsymbol{k}}}\; e^{-ik.(y-x)}\Big|_{k^0=\omega_{\boldsymbol{k}}} \\ &= \int \frac{d^3\boldsymbol{k}}{(2\pi)^3\, 2\omega_{\boldsymbol{k}}}\; e^{ik.(x-y)}\Big|_{k^0=\omega_{\boldsymbol{k}}}. \qquad (2)\end{aligned}$$

In summary,

$$\Delta_\pm(x,y) = \int \frac{d^3\boldsymbol{k}}{(2\pi)^3\, 2\omega_{\boldsymbol{k}}}\; e^{\mp ik.(x-y)}\Big|_{k^0=\omega_{\boldsymbol{k}}}. \qquad (3)$$

The RHS of (3) might not appear Lorentz invariant but recall,

$$\int \frac{d^3\boldsymbol{k}}{2\omega_{\boldsymbol{k}}} = \int d^4k\; \delta\left(k^2 - m^2\right)\; \theta\left(k^0\right).$$

So in Lorentz covariant notation we can express the Wightman functions as

$$\Delta_{\pm}(x,y) = \int \frac{d^4k}{(2\pi)^3}\, \delta\left(k^2 - m^2\right)\, \theta\left(k^0\right)\, e^{\mp ik.(x-y)}.$$

It is evident from their definition that the Wightman functions obey the Klein-Gordon equation,

$$\left(\Box_x + m^2\right)\Delta_{\pm}(x,y) = 0.$$

1.1.1 Wightman function at spacelike separations

One can show that the Wightman function is non-vanishing when x and y are spacelike separated, i.e. when $(x-y)^2 < 0$. For such spacelike separated spacetime points, one can find an inertial frame where $x^0 = y^0$ i.e. they are simultaneously occurring events. In such a case, the expression for the Wightman function (1) becomes,

$$\begin{aligned}\Delta_+(x,y) = \Delta_+\left(\boldsymbol{x}-\boldsymbol{y}\right) &= \int \frac{d^3\boldsymbol{k}}{(2\pi)^3\, 2\omega_{\boldsymbol{k}}} e^{i\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{y})} \\ &= \frac{m}{4\pi^2 r} K_1(m\, r), \quad r = |\boldsymbol{x}-\boldsymbol{y}| \qquad (4) \\ &\sim \sqrt{\frac{\pi m r}{2}}\frac{e^{-mr}}{4\pi^2 r^2}. \qquad (5)\end{aligned}$$

So the Wightman function decays exponentially fast for spacelike separations - signals cannot propagate far outside the light-cone.

Homework: Prove (4) by performing the integration in the previous line around the cut starting at im to $i\infty$. (Hint: To perform the $\boldsymbol{k}$-integration, use spherical polar co-ordinates with the z-axis aligned with the vector $(\boldsymbol{x}-\boldsymbol{y})$). Then prove (5) where "$\sim$" implies "asymptotically goes as" i.e. the Wightman functions decays exponentially for spacelike separations i.e. outside the light-cone. $(10+2$ points)

This expression (4) for the Wightman function for spacelike separated points can be immediately generalized to that of an arbitrary reference frame by writing it in terms of Lorentz invariant quantities such as the proper distance (length), $s = \sqrt{-(x-y)^2}$:

$$\Delta_+(x,y) = \frac{m}{4\pi^2 s} K_1\left(m\, s\right). \qquad (6)$$

Similarly for timelike separations one can show that,

$$\Delta_+(x,y) = \frac{im}{8\,\pi\,\tau} H_1^{(2)}\left(m\,\tau\right). \qquad (7)$$

where $\tau = \sqrt{(x-y)^2}$ is the proper time.

Homework: Derive (7) by performing the integral (1). (3 points) [Hint: Since x

and y are timelike separated, it is best to go to an inertial frame where $\boldsymbol{x} = \boldsymbol{y}$ and where the proper time, $\tau = x^0 - y^0$. The perform the $\boldsymbol{k}-$integral in spherical polar coordinates using Mathematica. Finally you may need to use the identity,

$$K_1(i\,x) = -\frac{\pi}{2} H_1^{(2)}(x).$$

This integral actually only converges if τ has a tiny *negative* imaginary part, i.e. $\tau - i\epsilon$. Similarly for $\Delta_-(x,y)$ the integral converges only if $\tau \to \tau + i\epsilon$.]

1.1.2 Wightman function at lightlike separations: Lightcone singularity

Finally we consider the case when the points x and y are lightlike separated, i.e. y lies on the lightcone of x. There is no need to do this computation separately, one can simply deduce the behavior by approaching the lightcone from spacelike or timelike separations i.e. taking $s^2 \to 0$ limit of the Wightman function for spacelike separated points (6) or $\tau^2 \to 0$ limit of the Wightman function for timelike separated points (7). Either way we see that there is a branch point (square root branch point) singularity in the Wightman function (distribution) as $\sigma^2 \to 0$, $\sigma^2 = (x-y)^2$. For obvious reasons, this essential singularity is known as the light cone singularity of the Wightman function at null (or even coincident) separations.

1.2 Hadamard elementary function (distribution)

The Hadamard elementary function or the Hadamard distribution is defined by the symmetric combination,

$$\begin{aligned}
\Delta^{(1)}(x,y) = \langle 0 | \{\varphi(x), \varphi(y)\} | 0 \rangle &= \Delta_+(x,y) + \Delta_-(x,y) \\
&= \int \frac{d^3\boldsymbol{k}}{(2\pi)^3\, 2\omega_{\boldsymbol{k}}} \left(e^{-ik.(x-y)} + e^{ik.(x-y)} \right)_{k^0=\omega_{\boldsymbol{k}}} \\
&= \int \frac{d^3\boldsymbol{k}}{(2\pi)^3\, \omega_{\boldsymbol{k}}} \cos\left(k.\,(x-y)\right)\Big|_{k^0=\omega_{\boldsymbol{k}}}.
\end{aligned}$$

In covariant notation,

$$\Delta^{(1)}(x,y) = \int \frac{d^4k}{(2\pi)^3}\, \delta\left(k^2 - m^2\right)\ e^{-ik.(x-y)}.$$

By construction, it is symmetric in the spacetime argument,

$$\Delta^{(1)}(x,y) = \Delta^{(1)}(y,x).$$

It is also evident that this is non-vanishing for spacelike separations i.e. when y is outside the light-cone of x just as $\Delta_\pm(x,y)$ are, and it also satisfies the Klein-Gordon equation,

$$\left(\Box_x + m^2\right) \Delta^{(1)}(x,y) = 0.$$

1.3 Schwinger function or Pauli-Jordan function

$$
\begin{aligned}
i\Delta(x,y) \equiv \langle 0 \big| \left[\varphi(x), \varphi(y)\right] \big| 0 \rangle &= \Delta_+(x,y) - \Delta_-(x,y) \\
&= \int \frac{d^3\boldsymbol{k}}{(2\pi)^3\, 2\omega_{\boldsymbol{k}}} \left(e^{-ik.(x-y)} - e^{ik.(x-y)}\right)_{k^0=\omega_{\boldsymbol{k}}}. \\
\Rightarrow \Delta(x,y) &= -\int \frac{d^3\boldsymbol{k}}{(2\pi)^3\, \omega_{\boldsymbol{k}}} \sin\left(k.\left(x-y\right)\right)\Big|_{k^0=\omega_{\boldsymbol{k}}}. \qquad (8)
\end{aligned}
$$

Evidently when $x^0 = y^0$ this vanishes as it is an integral of an odd function of $\boldsymbol{k}$ over $(-\infty, \infty)$. This is expected because according the canonical commutation relations

$$[\varphi(x), \varphi(y)]_{x^0=y^0} = 0.$$

Only in the free theory $[\varphi(x), \varphi(y)]$ is a c-number. In a general interacting theory $[\varphi(x), \varphi(y)]$ is an quantum operator i.e. q-number. Since it vanishes when $x^0 = y^0$, it immediately follows that it vanishes for spacelike separations $(x-y)^2 < 0$,

$$[\varphi(x), \varphi(y)]_{(x-y)^2<0} = 0.$$

(**Homework: Prove this statement**). This also makes sense, since for spacelike separations no causal signal (moving with speed less than or equal to that of light) can be sent from x to y, and hence the operators $\varphi(x)$ and $\varphi(x)$ represent independent measurements.

In covariant form,

$$\Delta(x,y) = \frac{1}{i} \int \frac{d^4k}{(2\pi)^3}\, \delta\left(k^2 - m^2\right)\, \epsilon\left(k^0\right)\, e^{-ik.(x-y)},$$

where $\epsilon(k^0)$ is the sign function,

$$\epsilon\left(k^0\right) = \begin{cases} +1, & k^0 > 0 \\ 0, & k = 0 \\ -1, & k^0 < 0. \end{cases}$$

$\Delta(x,y)$ too satisfies the Klein-Gordon equation,

$$\left(\Box_x + m^2\right)\Delta(x,y) = 0.$$

Clearly the Schwinger function is antisymmetric in the spacetime argument,

$$\Delta(x,y) = -\Delta(y,x).$$