PH6418/ PH4618: Quantum Field Theory (Spring 2022)

Notes for Lecture 17: Green's functions Klein-Gordon equation (Retarded, Advanced, Feynman time-ordered and anti-time-ordered)*

April 28, 2022

1 Recap of lecture 16 with comments:

1.1 Correlation functions in free scalar field theory

The result of any measurement involving the scalar field theory can be in terms of vacuum expectation values of a string of say $n$field operators at $n$different points in spacetime,

$$\langle 0|\varphi(x_1)\,\varphi(x_2)\ldots\varphi(x_n)|0\rangle$$

for arbitrary n. These are also called n-point correlation functions or simply n-point functions. These are the fundamental quantities to be computed on the theory side to facilitate comparison with experiments. Let's compute some low order correlation functions in the free scalar field theory. Even before we compute anything, just on the basis of the (discrete, internal) sign reflection symmetry,

$$\varphi(x)\rightarrow-\varphi(x),\quad\forall x,$$

we can say that *odd order correlation functions vanish.*

$$\langle 0|\varphi(x_1)\,\varphi(x_2)\ldots\varphi(x_n)|0\rangle=0,\quad n=\text{odd}.$$

The proof is as follows. Under the sign reflection symmetry all physical observables including the correlation functions must remain unchanged. Under sign reflection symmetry,

$$\langle 0|\varphi(x_1)\,\varphi(x_2)\ldots\varphi(x_n)|0\rangle\rightarrow(-)^n\,\langle 0|\varphi(x_1)\,\varphi(x_2)\ldots\varphi(x_n)|0\rangle$$

For odd n,

$$\langle 0|\varphi(x_1)\,\varphi(x_2)\ldots\varphi(x_n)|0\rangle=-\langle 0|\varphi(x_1)\,\varphi(x_2)\ldots\varphi(x_n)|0\rangle,$$

which means,

$$\langle 0|\varphi(x_1)\,\varphi(x_2)\ldots\varphi(x_n)|0\rangle=0,\quad n=\text{odd}.$$

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So the first nontrivial thing to compute are the various two point correlation functions,

$$\langle 0 \big| \varphi(x)\, \varphi(y) \big| 0 \rangle.$$

Since in quantum mechanics while taking products of two operators the ordering of operators is important, we will work out each case in some detail in the following sections.

1.2 Wightman functions

The Wightman functions are defined to be,

$$\Delta_+(x,y) = \langle 0 \big| \varphi(x)\varphi(y) \big| 0 \rangle,$$
$$\Delta_-(x,y) = \langle 0 \big| \varphi(y)\varphi(x) \big| 0 \rangle.$$

Expressed as a mode sum the Wightman function is

$$\Delta_\pm(x,y) = \int \frac{d^3\boldsymbol{k}}{(2\pi)^3\, 2\omega_{\boldsymbol{k}}}\; e^{\mp ik.(x-y)}\big|_{k^0=\omega_{\boldsymbol{k}}}\,. \tag{1}$$

Then the reverse ordered Wightman function is,

$$\Delta_-(x,y) = \Delta_+^\dagger(y,x)$$

Wightman functions in manifestly Lorentz invariant mode sum:

$$\Delta_\pm(x,y) = \int \frac{d^4k}{(2\pi)^3}\, \delta\left(k^2 - m^2\right)\; \theta\left(k^0\right)\; e^{\mp ik.(x-y)}.$$

It is evident from their definition that the Wightman functions obey the Klein-Gordon equation,

$$\left(\Box_x + m^2\right) \Delta_\pm(x,y) = 0.$$

1.2.1 Wightman function at spacelike separations

One can show that the Wightman function is non-vanishing when x and y are spacelike separated, i.e. when $(x-y)^2 < 0$. For such spacelike separated spacetime points, the Wightman function is,

$$\Delta_+(x,y) = \frac{m}{4\pi^2 s} K_1\left(m\, s\right) \sim \sqrt{\frac{\pi m s}{2}}\, \frac{e^{-ms}}{4\pi^2 s^2}, \tag{2}$$

where,

$$s \equiv \sqrt{-\left(x-y\right)^2}$$

is the proper distance between x and y. The Wightman function decays exponentially fast for spacelike separations, so signals cannot propagate far outside the light-cone.

Comment: The state $|\boldsymbol{x}, t\rangle \equiv \varphi(x)\,|0\rangle$ can be thought of as a state of a point particle at location $\boldsymbol{x}$ at time t but with caveats! This is because the amplitude,

$$\langle \boldsymbol{x}', t \,|\, \boldsymbol{x}, t\rangle \neq \delta^3\left(\boldsymbol{x}' - \boldsymbol{x}\right).$$

Instead,

$$\langle \boldsymbol{x}', t \,|\, \boldsymbol{x}, t\rangle = \frac{m}{4\pi^2 r} K_1\left(m\, r\right), \quad r = \left|\boldsymbol{x}' - \boldsymbol{x}\right|.$$

So $|\boldsymbol{x}, t\rangle \equiv \varphi(x)\,|0\rangle$ represents a point particle which is ***not sharply localized at $\boldsymbol{x}$, but instead smeared out over an extended region of dimensions $1/m$ centered around $\boldsymbol{x}$.***

Using the relativistic uncertainty principle (energy time uncertainty principle plus mass-energy equivalence),

$$\Delta x\, \Delta m > 1$$

one has,

$$\Delta m > \frac{1}{\Delta x}.$$

When $\Delta x < m^{-1}$, the Compton wavelength, then

$$\Delta m > m.$$

Thus the uncertainty in mass measurement confined in a region with dimensions shorter than of the Compton wavelength is more than the mass of a single particle! So we cannot tell if there is a single particle or multiple particles contained inside a region of size less than the Compton wavelength.

1.2.2 Wightman function at timelike separations

Similarly for timelike separations one can show that,

$$\Delta_+(x, y) = \frac{im}{8\,\pi\,\tau} H_1^{(2)}\left(m\,\tau\right). \tag{3}$$

where $\tau = \sqrt{(x-y)^2}$ is the proper time.

The integral expression (1) for $\Delta_+(x, y)$, only converges if τ has a tiny *negative* imaginary part, i.e. $\tau - i\epsilon$. Similarly for $\Delta_-(x, y)$ the integral converges only if $\tau \to \tau + i\epsilon$.]

1.2.3 Wightman function at lightlike separations: Lightcone singularity

Approaching the lightcone from spacelike or timelike separations i.e. taking $\sigma = (x-y)^2 \to 0$ limit of the Wightman function for timelike separated points (3) or for timelike separated points (2), we see that there is a branch point singularity (square root branch point) in the Wightman function (distribution) as $\sigma \to 0$. For obvious reasons, this essential singularity is known as the light cone singularity of the Wightman function at lightlike (or coincident) separations.

1.3 Hadamard elementary function (distribution) and Schwinger function or Pauli-Jordan function

The Hadamard elementary function or the Hadamard distribution and the Schwinger function (or Pauli-Jordan function) are defined by the combinations,

$$\begin{aligned} \Delta^{(1)}(x, y) &= \langle 0|\left\{\varphi(x), \varphi(y)\right\}|0\rangle = \Delta_+(x, y) + \Delta_-(x, y) \\ i\Delta(x, y) &\equiv \langle 0|\left[\varphi(x), \varphi(y)\right]|0\rangle = \Delta_+(x, y) - \Delta_-(x, y) \end{aligned}$$

By construction, they are respectively symmetric and antisymmetric in the spacetime argument,

$$\Delta^{(1)}(x,y) = \Delta^{(1)}(y,x),$$
$$\Delta(x,y) = -\Delta(y,x)$$

Expressed as mode sums,

$$\Delta^{(1)}(x,y) = \int \frac{d^3\boldsymbol{k}}{(2\pi)^3\,\omega_{\boldsymbol{k}}} \cos\left(k.\left(x-y\right)\right)\big|_{k^0=\omega_{\boldsymbol{k}}},$$
$$\Delta(x,y) = -\int \frac{d^3\boldsymbol{k}}{(2\pi)^3\,\omega_{\boldsymbol{k}}} \sin\left(k.\left(x-y\right)\right)\big|_{k^0=\omega_{\boldsymbol{k}}}.$$

When $x^0 = y^0$ the Schwinger function vanishes. One can find a Lorentz frame where, $x^0 = y^0$,and the mode sum is an integral of an odd function of $\boldsymbol{k}$ over $(-\infty,\infty)$. This is expected because according the canonical commutation relations

$$[\varphi(x),\varphi(y)]_{x^0=y^0} = 0.$$

Comment: Only in the free theory $[\varphi(x),\varphi(y)]$ is a c-number. In a general interacting theory $[\varphi(x),\varphi(y)]$ is an quantum operator i.e. q-number. Since it vanishes when $x^0 = y^0$, it immediately follows that it vanishes for spacelike separations $(x-y)^2 < 0$,

$$[\varphi(x),\varphi(y)]_{(x-y)^2<0} = 0.$$

This also makes sense, since for spacelike separations no causal signal (moving with speed less than or equal to that of light) can be sent from x to y, and hence the operators $\varphi(x)$ and $\varphi(x)$ represent independent measurements.

2 Feynman Propagator (Time-ordered correlation function)

This is defined by the time-ordered correlation function of two field operators:

$$i\Delta_F(x,y) \equiv \langle 0\big|T\left(\varphi(x)\varphi(y)\right)\big|0\rangle \equiv \theta\left(x^0-y^0\right)\ \langle 0\big|\varphi(x)\varphi(y)\big|0\rangle + \theta\left(y^0-x^0\right)\ \langle 0\big|\varphi(y)\varphi(x)\big|0\rangle.$$

In terms of the Wightman functions,

$$\begin{aligned} i\Delta_F(x,y) &= \theta\left(x^0-y^0\right)\ \Delta_+(x,y) + \theta\left(y^0-x^0\right)\ \Delta_-(x,y) \\ &= \theta\left(x^0-y^0\right)\int \frac{d^3\boldsymbol{k}}{(2\pi)^3\,2\omega_{\boldsymbol{k}}}\, e^{-ik.(x-y)}\big|_{k^0=\omega_{\boldsymbol{k}}} + \theta\left(y^0-x^0\right)\int \frac{d^3\boldsymbol{k}}{(2\pi)^3\,2\omega_{\boldsymbol{k}}}\, e^{ik.(x-y)}\big|_{k^0=\omega_{\boldsymbol{k}}}. \end{aligned} \tag{4}$$

- Evidently it is symmetric in the spacetime argument,

$$\Delta_F(x,y) = \Delta_F(y,x).$$

- Physical interpretation - "First create, then destroy": Let's say$x^0 > y^0$, then

$$i\Delta_F(x,y) = \langle 0\big|\varphi(x)\varphi(y)\big|0\rangle$$

This can be thought of as the probability amplitude for the process of getting to vacuum, $\langle 0\big|$ from the state/ket $\varphi(x)\varphi(y)\big|0\rangle$. Since $\varphi(y)$ creates a particle out of the vacuum at spacetime location y, it needs to be annihilated at the spacetime point x in the future to get to the vacuum. Thus we "first create, and then destroy". Another way we can look at this is the overlap of the one-particle state $\big|y\rangle = \varphi(y)\big|0\rangle$ in the past with the one-particle state $\big|x\rangle = \varphi(x)|0\rangle$ in the future, i.e the probability amplitude for a particle initially at $\boldsymbol{y}$ at time y^0 to be at $\boldsymbol{x}$ at a later time x^0,

$$\langle x\big|y\rangle$$

Thus this represents the probability amplitude of a particle propagating from $\boldsymbol{y}$ at time y^0 in the past to the location $\boldsymbol{x}$ at future time x^0.

- It is a Green's function to Klein-Gordon equation:

$$\left(\Box_x + m^2\right)\Delta_F(x,y) = -\delta^4(x-y).$$

Homework: Show this.

3 Green's functions for Klein-Gordon equation

The Klein-Gordon equation with a source $j(x)$ i.e.

$$\left(\Box_x + m^2\right)\varphi(x) = j(x),$$

can be solved by using a Green's function, $G(x,y)$

$$\varphi(x) = -\int d^4y\, G(x,y)\, j(y),$$

which satisfies the equation,

$$\left(\Box_x + m^2\right)G(x,y) = -\delta^4(x-y), \tag{5}$$

The Green's function is not unique, it is only fully determined after specifying boundary conditions.

Translation symmetry implies,

$$G(x,y) = G(x-y),$$

i.e. it is a scalar function of a single variable (4-vector), namely $(x-y)$ and not two independent variables, 4-vectors. To solve for the Klein-Gordon Green's function i.e. (5), we go to Fourier space

$$G(x-y) = \int \frac{d^4k}{(2\pi)^4} e^{-ik.(x-y)} G(k), \tag{6}$$

and substitute in (5). In Fourier space, the equation (5) turns into an algebraic equation,

$$\left(-k^2 + m^2\right)G(k) = -1,$$

or,

$$G(k) = \frac{1}{k^2 - m^2}. \tag{7}$$

Now returning to position space,

$$\begin{aligned} G(x,y) &= \int \frac{d^4k}{(2\pi)^4} e^{-ik.(x-y)} \frac{1}{k^2 - m^2} \\ &= \int \frac{d^4k}{(2\pi)^4} e^{-ik.(x-y)} \frac{1}{(k^0)^2 - \boldsymbol{k}\cdot\boldsymbol{k} - m^2} \\ &= \int \frac{d^4k}{(2\pi)^4} e^{-ik.(x-y)} \frac{1}{(k^0)^2 - \omega_{\boldsymbol{k}}^2}, \qquad \omega_{\boldsymbol{k}} = \sqrt{\boldsymbol{k}^2 + m^2} \\ &= \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})} \left(\int \frac{dk^0}{2\pi} \frac{e^{-ik^0\left(x^0-y^0\right)}}{(k^0)^2 - \omega_{\boldsymbol{k}}^2} \right). \end{aligned} \tag{8}$$

At this point if we assume that the k^0-integration is to be performed on the real line $(-\infty, \infty)$, then we will be unable to perform the Fourier inversion integral for k^0 because there poles on the real line contour at $k^0 = \pm\omega_{\boldsymbol{k}}$. However in the Fourier transform definition (6) we never specified what the contour of integration is, the only requirement is that the transform exist and be real valued. Keeping this in mind we will deform the contour of integration off the real k^0 line into the complex k^0-plane or equivalently adopt a suitable ***pole prescription*** which will move the poles at $k^0 = \pm\omega_{\boldsymbol{k}}$ off the real k^0-axis by endowing them infinitesimal imaginary parts. In what follows we will choose pole prescriptions which will be determined by the choice of boundary conditions for the Green's function, $G(x, y)$.

3.1 Classical Green's function: Retarded and Advanced Propagators

The retarded propagator or retarded Green's function, $\Delta_R(x, y)$ is defined to be the Green's function which ***solely*** propagates signal (or data or information) forward in time, i.e. from past to future. Mathematically,

$$\Delta_R(x,y) \propto \theta\left(x^0 - y^0\right).$$

The second argument, y is the spacetime point which is in the past while, the first argument x is in the future. Now let's return to the Fourier integral representation of the Green's function (8) and see which pole prescription gets selected as a consequence of this boundary condition. In particular let's concentrate on the k^0-integral, namely,

$$I = \int_{-\infty}^{\infty} \frac{dk^0}{2\pi} \frac{e^{-ik^0\left(x^0-y^0\right)}}{(k^0)^2 - \omega_{\boldsymbol{k}}^2}.$$

We will think of this integral not as a real integral i.e. over the real line but as a complex integral over a contour on the complex k^0-plane, and in particular not an open contour but a *closed* contour

$$J = \oint \frac{dk^0}{2\pi} \frac{e^{-ik^0\left(x^0-y^0\right)}}{(k^0)^2 - \omega_{\boldsymbol{k}}^2} \tag{9}$$

and then use the Cauchy residue theorem to evaluate it. The closed contour must be chosen such that part of it coincides with the real line $(-\infty, \infty)$, while the rest of it i.e. back from real $+\infty$ to real $-\infty$ is off the real line via the complex k^0-plane via some contour $\mathcal{C}$

$$J = I + \int_{\mathcal{C}\infty}^{-\infty} \frac{dk^0}{2\pi} \frac{e^{-ik^0\left(x^0-y^0\right)}}{\left(k^0\right)^2 - \omega_{\boldsymbol{k}}^2}$$

This second (return) part of the contour will be determined so that it makes a vanishing contribution to the contour integral,

$$\int_{\mathcal{C}\infty}^{-\infty} \frac{dk^0}{2\pi} \frac{e^{-ik^0\left(x^0-y^0\right)}}{\left(k^0\right)^2 - \omega_{\boldsymbol{k}}^2}, \tag{10}$$

so that the complex integral gives the same answer as the original real integral:

$$J_0 = I_0.$$

Now for $x^0 > y^0$ the integral J_0 is damped in the lower half k^0-plane (LHP) because of the exponential piece:

$$e^{-ik^0\left(x^0-y^0\right)} = e^{-i\left(\mathrm{Re}k^0 + i\mathrm{Im}k^0\right)\left(x^0-y^0\right)} = e^{-i\,\mathrm{Re}k^0\left(x^0-y^0\right)} e^{-i\mathrm{Im}k^0\left(x^0-y^0\right)}.$$

In the LHP, when $\mathrm{Im}k^0$ is large and negative then for $\left(x^0 - y^0\right)$ this provides a large damping factor $e^{-i\mathrm{Im}k^0\left(x^0-y^0\right)}$. On the other hand when $x^0 < y^0$, the exponential term is damped in the upper half k^0-plane (UHP). So the contour integral (10) will make a vanishing contribution when it is closed in the LHP for $x^0 > y^0$ and it will make vanishing contribution if it is closed in the UHP for $x^0 < y^0$.

$$J = \theta\left(x^0 - y^0\right) \oint \frac{dk^0}{2\pi} \frac{e^{-ik^0\left(x^0-y^0\right)}}{\left(k^0\right)^2 - \omega_{\boldsymbol{k}}^2} + \theta\left(y^0 - x^0\right) \oint \frac{dk^0}{2\pi} \frac{e^{-ik^0\left(x^0-y^0\right)}}{\left(k^0\right)^2 - \omega_{\boldsymbol{k}}^2}$$

It is evident that the contour has to be traversed in the clockwise (counterclockwise) direction to be closed in the LHP (UHP). Then it is obvious that for the retarded Green's function we will need a pole prescription such that the poles are only enclosed by the contour closed in the LHP, i.e. shifts *both* the poles $k^0 = \pm\omega_{\boldsymbol{k}}$ off the real axis into the LHP, to wit, $k^0 = \pm\omega_{\boldsymbol{k}} - i\varepsilon$. (Refer to the contour on the left in figure 1). Here ε is an infinitesimal positive quantity which will be taken to zero at a later stage. The residue theorem will then yield a non-vanishing result for $x^0 > y^0$ and a vanishing result for $x^0 < y^0$. Since for the retarded propagator, the poles are located at $k^0 = \pm\omega_{\boldsymbol{k}} - i\varepsilon$ or at $\left(k^0 + i\varepsilon\right)^2 - \omega_{\boldsymbol{k}}^2 = 0$, we will make the replacement the denominator of (9),

$$\begin{aligned} J_R &= \theta\left(x^0 - y^0\right) \oint \frac{dk^0}{2\pi} \frac{e^{-ik^0\left(x^0-y^0\right)}}{\left(k^0 + i\varepsilon\right)^2 - \omega_{\boldsymbol{k}}^2} + \underbrace{\theta\left(y^0 - x^0\right) \oint \frac{dk^0}{2\pi} \frac{e^{-ik^0\left(x^0-y^0\right)}}{\left(k^0 + i\varepsilon\right)^2 - \omega_{\boldsymbol{k}}^2}}_{=0} \\ &= \theta\left(x^0 - y^0\right) \oint \frac{dk^0}{2\pi} \frac{e^{-ik^0\left(x^0-y^0\right)}}{\left(k^0 + i\varepsilon\right)^2 - \omega_{\boldsymbol{k}}^2} \\ &= \theta\left(x^0 - y^0\right) \left[-i\left(\frac{e^{-i(\omega_{\boldsymbol{k}} - i\varepsilon)\left(x^0-y^0\right)}}{2\left(\omega_{\boldsymbol{k}} - i\varepsilon\right)} + \frac{e^{-i(-\omega_{\boldsymbol{k}} - i\varepsilon)\left(x^0-y^0\right)}}{-2\left(\omega_{\boldsymbol{k}} + i\varepsilon\right)}\right)\right]. \end{aligned}$$

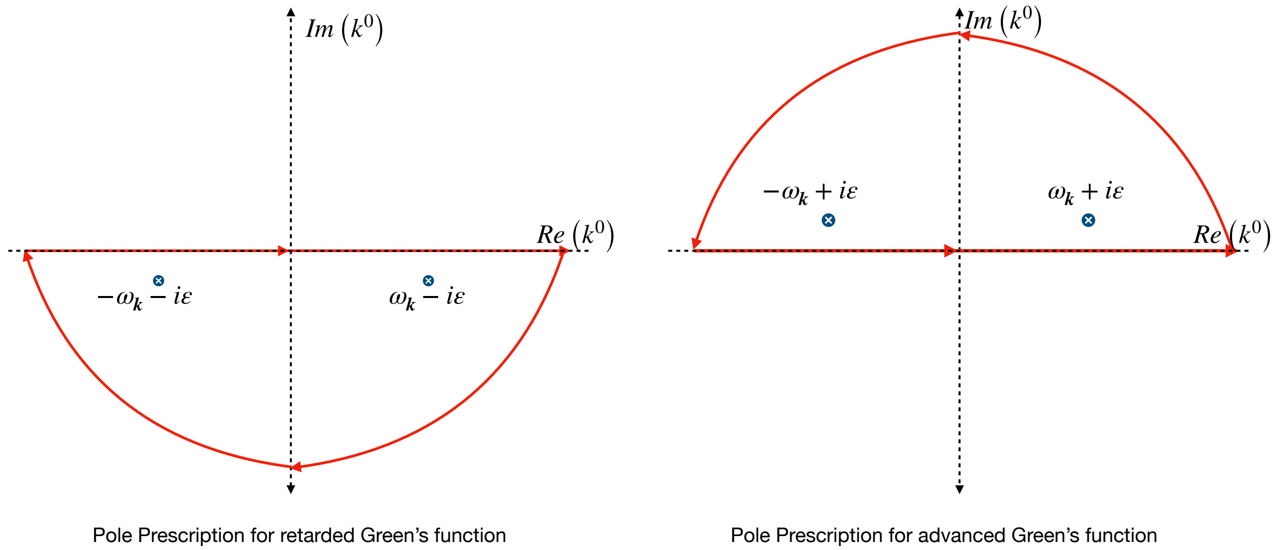

Figure 1: Pole prescriptions for retarded and advanced Green's functions

At this point it is safe to take the limit $\varepsilon \to 0$, and we have,

$$J_R = \frac{1}{i} \frac{\theta\left(x^0 - y^0\right)}{2\omega_{\boldsymbol{k}}} \left(e^{-i\omega_{\boldsymbol{k}}\left(x^0-y^0\right)} - e^{i\omega_{\boldsymbol{k}}\left(x^0-y^0\right)}\right) \tag{11}$$

Plugging this back in Fourier integral formula (8), we get,

$$\begin{aligned}
\Delta_R(x,y) &= \theta\left(x^0-y^0\right)\frac{1}{i}\int \frac{d^3\boldsymbol{k}}{(2\pi)^3\, 2\omega_{\boldsymbol{k}}} e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})}\left(e^{-i\omega_{\boldsymbol{k}}\left(x^0-y^0\right)} - e^{i\omega_{\boldsymbol{k}}\left(x^0-y^0\right)}\right) \\
&= \theta\left(x^0-y^0\right)\frac{1}{i}\left(\int \frac{d^3\boldsymbol{k}}{(2\pi)^3\, 2\omega_{\boldsymbol{k}}} e^{-i\left[\omega_{\boldsymbol{k}}\left(x^0-y^0\right)-\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})\right]} - \underbrace{\int \frac{d^3\boldsymbol{k}}{(2\pi)^3} e^{i\left[\omega_{\boldsymbol{k}}\left(x^0-y^0\right)+\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})\right]}}_{\boldsymbol{k}\to-\boldsymbol{k}}\right) \\
&= \theta\left(x^0-y^0\right)\frac{1}{i}\left(\int \frac{d^3\boldsymbol{k}}{(2\pi)^3\, 2\omega_{\boldsymbol{k}}} e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})} e^{-i\omega_{\boldsymbol{k}}\left(x^0-y^0\right)} - \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} e^{i\left[\omega_{\boldsymbol{k}}\left(x^0-y^0\right)-\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})\right]}\right) \\
&= \theta\left(x^0-y^0\right)\frac{1}{i}\int \frac{d^3\boldsymbol{k}}{(2\pi)^3\, 2\omega_{\boldsymbol{k}}}\left(e^{-ik.(x-y)} - e^{ik.(x-y)}\right)_{k^0=\omega_{\boldsymbol{k}}} \\
&= -\theta\left(x^0-y^0\right)\int \frac{d^3\boldsymbol{k}}{(2\pi)^3\, \omega_{\boldsymbol{k}}} \sin\left(k.\,(x-y)\right)\big|_{k^0=\omega_{\boldsymbol{k}}}\,.
\end{aligned} \tag{12}$$

Evidently the retarded Green's function is proportional to the Schwinger function, $\Delta(x,y) = -\int \frac{d^3\boldsymbol{k}}{(2\pi)^3\omega_{\boldsymbol{k}}} \sin\left(k.\,(x-y)\right)\big|_{k^0=\omega_{\boldsymbol{k}}}$,

$$\Delta_R(x,y) = \theta\left(x^0-y^0\right)\Delta(x,y). \tag{13}$$

Hence the retarded Green's function has support when the point y lies inside the past light cone of x, i.e. $\Delta_R(x,y)$ is non-vanishing if and only if $(x-y)^2 > 0$ and $x^0 > y^0$. This is because the Schwinger function has support only inside lightcones, i.e. when $(x-y)^2 > 0$, while the step function $\theta(x^0 - y^0)$ has support only when y is to the past of x - these conditions are satisfied together only when y lies inside the past light cone of x.

In covariant notation the pole prescription for the retarded propagator can be written as,

$$\Delta_R(k) = \frac{1}{\left(k^0 + i\varepsilon\right)^2 - \omega_{\boldsymbol{k}}^2} = \frac{1}{k^2 - m^2 + i\,\theta\left(k^0\right)\varepsilon}.$$

One can also define an advanced Green's function, $\Delta_A(x,y)$ which recovers present data (or information or the field configuration) from its future data[1]. The spacetime point for the first argument x is the present while the second argument, y is in the future. Since this propagator selectively recovers present data from future data, it must vanish when $x^0 > y^0$ and nonvanishing when $x^0 < y^0$, i.e.

$$\Delta_A(x,y) \propto \theta\left(y^0 - x^0\right).$$

The pole prescription that is appropriate for this boundary condition is

$$k^0 = \pm\omega_{\boldsymbol{k}} + i\varepsilon,$$

(Refer to the contour on the right in figure 1), or in covariant notation,

$$\Delta_A(k) = \frac{1}{\left(k^0 - i\varepsilon\right)^2 - \omega_{\boldsymbol{k}}^2} = \frac{1}{k^2 - m^2 - i\,\theta\left(k^0\right)\varepsilon}.$$

Using this pole prescription one can work out the expression of the advanced Green's function as the Fourier integral,

$$\Delta_A(x,y) = \theta\left(y^0 - x^0\right) \int \frac{d^3\boldsymbol{k}}{(2\pi)^3\,\omega_{\boldsymbol{k}}} \sin\left(k.\left(x-y\right)\right)\Big|_{k^0=\omega_{\boldsymbol{k}}} = -\theta\left(y^0 - x^0\right)\Delta(x,y). \tag{14}$$

Not only does this Green's function have nonvanishing support when the point y in the future of x i.e. when $x^0 < y^0$ but in fact when the point y lies inside the the future lightcone of x: $\left\{y : (x-y)^2 < 0 \cap \theta\left(y^0 > x^0\right)\right\}$.

The retarded and advanced Green's functions are causal Green's functions (since they are supported inside either the past or the future lightcone) which arise in classical contexts and are familiar to most students from their electrodynamics courses. However in quantum field theory contexts there arises a third Green's function, namely the Feynman Green's function which appears in calculations of S-matrix elements. For quantum field theories at finite temperature one still needs a fourth Green's function, namely the anti-Feynman Green's function. We work them out in detail in the following sections.

[1] This does not violate causality because one is merely asking the question, given a field configuration in some future time, which present time field configuration did it evolve from.

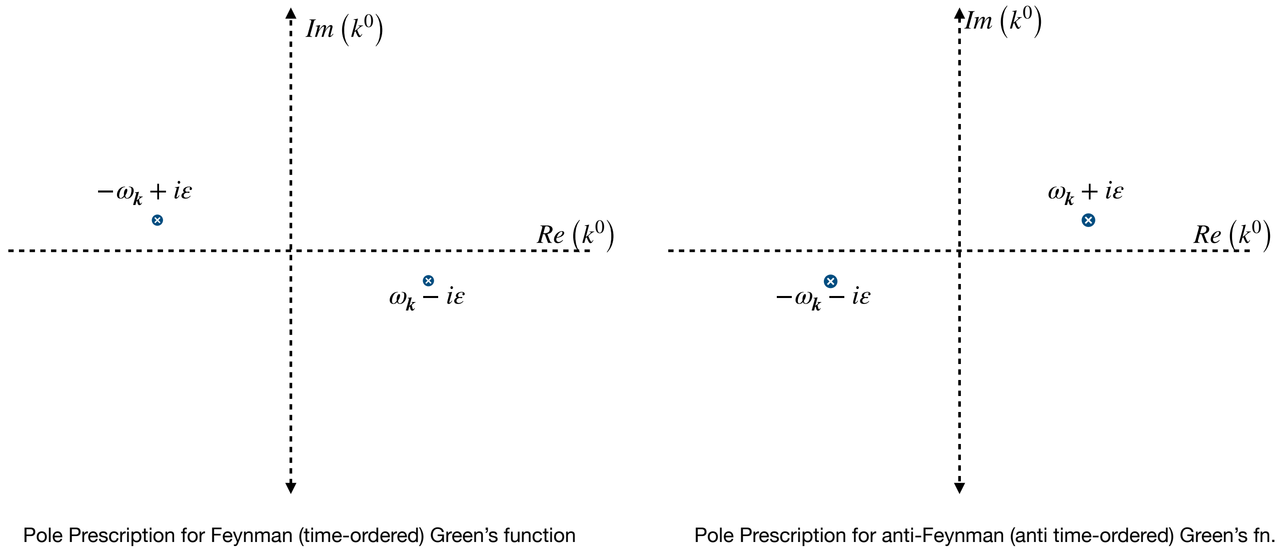

Figure 2: Pole prescriptions for Feynman (time-ordered) and anti-Feynman (anti-time-ordered) Green's functions

3.2 Feynman Propagator or Time-ordered Green's function

In the case of the classical causal Green's functions the pole prescriptions were such that both poles were either shifted to the LHP (retarded) or both poles were shifted into the UHP (advanced) i.e. ***both poles were shifted to the same side of the real k^0 axis***. This begs the question what kind of Green's functions does one obtain if the poles are shifted towards ***different sides of the real k^0-axis***. There are two cases to consider. First,

$$k^0 = \pm\omega_{\boldsymbol{k}} \mp i\varepsilon, \Leftrightarrow \frac{1}{\left(k^0\right)^2 - \omega_{\boldsymbol{k}}^2} \to \frac{1}{\left(k^0\right)^2 - \left(\omega_{\boldsymbol{k}} - i\varepsilon\right)^2}. \tag{15}$$

and the second

$$k^0 = \pm\omega_{\boldsymbol{k}} \pm i\varepsilon, \Leftrightarrow \frac{1}{\left(k^0\right)^2 - \omega_{\boldsymbol{k}}^2} \to \frac{1}{\left(k^0\right)^2 - \left(\omega_{\boldsymbol{k}} + i\varepsilon\right)^2}. \tag{16}$$

These are indicated in the figure 2. For the first case the k^0-integral, call it J_F is,

$$\begin{aligned} J_F &= \theta\left(x^0 - y^0\right) \oint \frac{dk^0}{2\pi} \frac{e^{-ik^0\left(x^0-y^0\right)}}{\left(k^0\right)^2 - \left(\omega_{\boldsymbol{k}} - i\varepsilon\right)^2} + \theta\left(y^0 - x^0\right) \oint \frac{dk^0}{2\pi} \frac{e^{-ik^0\left(x^0-y^0\right)}}{\left(k^0\right)^2 - \left(\omega_{\boldsymbol{k}} - i\varepsilon\right)^2} \\ &= -i\,\theta\left(x^0 - y^0\right) \frac{e^{-i\omega_{\boldsymbol{k}}\left(x^0-y^0\right)}}{2\omega_{\boldsymbol{k}}} - i\,\theta\left(y^0 - x^0\right) \frac{e^{i\omega_{\boldsymbol{k}}\left(x^0-y^0\right)}}{2\omega_{\boldsymbol{k}}}. \end{aligned}$$

Plugging this in the Fourier integral (8), we get a novel Green's function, call it G_F:

$$\begin{aligned}
G_F(x,y) &= -i \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})} \left(\theta\left(x^0-y^0\right) \frac{e^{-i\omega_{\boldsymbol{k}}\left(x^0-y^0\right)}}{2\omega_{\boldsymbol{k}}} + \theta\left(y^0-x^0\right) \frac{e^{i\omega_{\boldsymbol{k}}\left(x^0-y^0\right)}}{2\omega_{\boldsymbol{k}}} \right) \\
&= -i\,\theta\left(x^0-y^0\right) \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})} \frac{e^{-i\omega_{\boldsymbol{k}}\left(x^0-y^0\right)}}{2\omega_{\boldsymbol{k}}} - i\,\theta\left(y^0-x^0\right) \underbrace{\int \frac{d^3\boldsymbol{k}}{(2\pi)^3} e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})} \frac{e^{i\omega_{\boldsymbol{k}}\left(x^0-y^0\right)}}{2\omega_{\boldsymbol{k}}}}_{\boldsymbol{k}\to-\boldsymbol{k}} \\
&= \frac{1}{i}\left[\theta\left(x^0-y^0\right) \int \frac{d^3\boldsymbol{k}}{(2\pi)^3\, 2\omega_{\boldsymbol{k}}}\, e^{-ik.(x-y)}\Big|_{k^0=\omega_{\boldsymbol{k}}} + \theta\left(y^0-x^0\right) \int \frac{d^3\boldsymbol{k}}{(2\pi)^3\, 2\omega_{\boldsymbol{k}}}\, e^{ik.(x-y)}\Big|_{k^0=\omega_{\boldsymbol{k}}} \right].
\end{aligned} \tag{17}$$

But lo and behold, this expression is exactly same as that of the Feynman propagator, $\Delta_F(x,y)$, Eq. (4) which was introduced in Sec. 2 in terms of the time-ordered two point correlation function $\langle 0 | T\left(\varphi(x)\varphi(y)\right) | 0 \rangle$. Evidently, the Feynman Green's function, $G_F(x,y)$ has support for the point y inside ***both the past and future lightcones*** of the point x as well as outside the lightcones i.e. it is non-vanishing at spacelike separations but it falls off rapidly $\sim e^{-mr}, r = |\boldsymbol{x}-\boldsymbol{y}|$.

Finally the second choice of pole-prescription (16), dubbed the anti-Feynman Green's function,

$$\Delta_{\bar{F}}(x,y) = \frac{1}{i} \langle 0 | \bar{T}\left(\varphi(x)\varphi(y)\right) | 0 \rangle,$$

where $\bar{T}$ denotes anti-time ordering of two operators is defined as follows,

$$\bar{T}\left(\varphi(x)\varphi(y)\right) \equiv \theta\left(y^0-x^0\right)\varphi(x)\varphi(y) + \theta\left(x^0-y^0\right)\varphi(y)\varphi(x).$$

Again this has nonvanishing support for y lying in both the past and future lightcones of x as well for spacelike separated x and y.

Incidentally the pole prescriptions (15), (16) can be expressed in a manifestly Lorentz invariant notation,

$$\begin{aligned}
G_F(k) &= \frac{1}{k^2-m^2+i\varepsilon}, \\
G_{\bar{F}}(k) &= \frac{1}{k^2-m^2-i\varepsilon}.
\end{aligned}$$