Quantum Field Theory (PH-6418/ EP-4618): Final Exam

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Instructions: All questions are compulsory. This is an open notes exam - you may use class notes or lecture notes, formula sheet etc. but use of books and online resources are not allowed. Maximum score is 50, and the duration is 3 hours.

1. Show that the Feynman propagator for the free real quantum scalar field defined by

$$i\Delta_F(x,y) \equiv \theta\left(x^0 - y^0\right)\,\langle 0|\,\varphi(x)\,\varphi(y)\,|0\rangle + \theta\left(y^0 - x^0\right)\,\langle 0\,\varphi(y)\,\varphi(x)\,|0\rangle$$

satisfies the Green's function equation,

$$\left(\Box_x + m^2\right)\,\Delta_F(x,y) = -\delta^4(x-y).$$

(5 points)

Solution: Taking a time-derivative of the Feynman propagator,

$$\begin{aligned}\partial_{x^0}\left(i\Delta(x-y)\right) &= \underbrace{\delta\left(x^0-y^0\right)\,\langle 0|\left[\varphi(x),\varphi(y)\right]|0\rangle}_{=0 \text{ vide ETCCR}} + \theta\left(x^0-y^0\right)\,\langle 0|\,\dot{\varphi}(x)\,\varphi(y)\,|0\rangle + \theta\left(y^0-x^0\right)\,\langle 0\,\varphi(y)\,\dot{\varphi}(x)\,|0\rangle \\ &= \theta\left(x^0-y^0\right)\,\langle 0|\,\dot{\varphi}(x)\,\varphi(y)\,|0\rangle + \theta\left(y^0-x^0\right)\,\langle 0\,\varphi(y)\,\dot{\varphi}(x)\,|0\rangle\,.\end{aligned}$$

Taking another time derivative,

$$\begin{aligned}\partial^2_{x^0}\left(i\Delta(x-y)\right) &= \underbrace{\delta\left(x^0-y^0\right)\,\langle 0|\left[\dot{\varphi}(x),\varphi(y)\right]|0\rangle}_{=-i\,\delta^4(x-y) \text{ vide ETCCR}} + \theta\left(x^0-y^0\right)\,\langle 0|\,\ddot{\varphi}(x)\,\varphi(y)\,|0\rangle + \theta\left(y^0-x^0\right)\,\langle 0\,\varphi(y)\,\ddot{\varphi}(x)\,|0\rangle \\ &= -i\,\delta^4(x-y) + \theta\left(x^0-y^0\right)\,\langle 0|\,\ddot{\varphi}(x)\,\varphi(y)\,|0\rangle + \theta\left(y^0-x^0\right)\,\langle 0\,\varphi(y)\,\ddot{\varphi}(x)\,|0\rangle\,. \qquad (1)\end{aligned}$$

Next,

$$\boldsymbol{\nabla}^2_x\left(i\Delta(x-y)\right) = \theta\left(x^0-y^0\right)\,\langle 0|\,\boldsymbol{\nabla}^2\varphi(x)\,\varphi(y)\,|0\rangle + \theta\left(y^0-x^0\right)\,\langle 0\,\varphi(y)\,\boldsymbol{\nabla}^2\varphi(x)\,|0\rangle\,. \qquad (2)$$

Subtracting (2) from (1) we get,

$$\Box^2_x\left(i\Delta(x-y)\right) = -i\,\delta^4(x-y) + \theta\left(x^0-y^0\right)\,\langle 0|\,\Box_x\varphi(x)\,\varphi(y)\,|0\rangle + \theta\left(y^0-x^0\right)\,\langle 0\,\varphi(y)\,\Box_x\varphi(x)\,|0\rangle\,.$$

Next, in the RHS of this equation, we use the equation of motion for the free field operator, namely, $\left(\Box_x + m^2\right)\varphi(x) = 0$ to replace,

$$\Box_x\varphi(x) = -m^2\varphi(x),$$

and obtain,

$$\Box^2_x\left(i\Delta(x-y)\right) = -i\,\delta^4(x-y) - m^2\,\underbrace{\left(\theta\left(x^0-y^0\right)\,\langle 0|\,\varphi(x)\,\varphi(y)\,|0\rangle + \theta\left(y^0-x^0\right)\,\langle 0\,\varphi(y)\,\varphi(x)\,|0\rangle\right)}_{=i\,\Delta_F(x-y)}$$

or,

$$\left(\Box_x + m^2\right) \left(i\Delta(x-y)\right) = -i\,\delta^4(x-y),$$

or,

$$\left(\Box_x + m^2\right) \Delta(x-y) = -\delta^4(x-y).$$

2. Show that the state in the free real quantum scalar field theory,

$$\left(a^\dagger(\boldsymbol{p}_1)\right)^{n_1} \ \left(a^\dagger(\boldsymbol{p}_2)\right)^{n_2} |0\rangle$$

can be interpreted as a state containing $n_1 + n_2$ relativistic particles of mass m, out of which n_1 have linear 3-momentum $\boldsymbol{p}_1$ and n_2 of them have linear 3-momentum $\boldsymbol{p}_2$. Hint: Check the excitation energy and momentum of this state compared to the vacuum state.

(5 points)

Solution: The Hamiltonian is,

$$H = E_0 + \int d^3\boldsymbol{k}\, \omega_{\boldsymbol{k}}\, a^\dagger(\boldsymbol{k})\, a(\boldsymbol{k})$$

while the momentum is,

$$\boldsymbol{P} = \int d^3\boldsymbol{k}\, \boldsymbol{k}\, a^\dagger(\boldsymbol{k})\, a(\boldsymbol{k}).$$

Here E_0 is the divergent vacuum energy. We first work out the following algebraic simplification,

$$\begin{aligned}
a^\dagger(\boldsymbol{k})\, a(\boldsymbol{k}) \left(a^\dagger(\boldsymbol{p}_1)\right)^{n_1} \ \left(a^\dagger(\boldsymbol{p}_2)\right)^{n_2} |0\rangle &= a^\dagger(\boldsymbol{k}) \ \left[a(\boldsymbol{k}), \left(a^\dagger(\boldsymbol{p}_1)\right)^{n_1} \ \left(a^\dagger(\boldsymbol{p}_2)\right)^{n_2}\right] |0\rangle + a^\dagger(\boldsymbol{k}) \ \left(a^\dagger(\boldsymbol{p}_1)\right)^{n_1} \ \left(a^\dagger(\boldsymbol{p}_2)\right)^{n_2} \cancelto{0}{a(\boldsymbol{k})|0\rangle} \\
&= a^\dagger(\boldsymbol{k}) \ \left[a(\boldsymbol{k}), \left(a^\dagger(\boldsymbol{p}_1)\right)^{n_1}\right] \ \left(a^\dagger(\boldsymbol{p}_2)\right)^{n_2} |0\rangle + a^\dagger(\boldsymbol{k}) \ \left(a^\dagger(\boldsymbol{p}_1)\right)^{n_1} \ \left[a(\boldsymbol{k}), \left(a^\dagger(\boldsymbol{p}_2)\right)^{n_2}\right] |0\rangle \\
&= n_1 a^\dagger(\boldsymbol{k})\, \delta(\boldsymbol{k}-\boldsymbol{p}_1) \left(a^\dagger(\boldsymbol{p}_1)\right)^{n_1-1} \ \left(a^\dagger(\boldsymbol{p}_2)\right)^{n_2} |0\rangle + a^\dagger(\boldsymbol{k}) \ \left(a^\dagger(\boldsymbol{p}_1)\right)^{n_1} \ n_2\delta^3(\boldsymbol{k}-\boldsymbol{p}_2) \left(a\right. \\
&= n_1\delta^3(\boldsymbol{k}-\boldsymbol{p}_1) \left(a^\dagger(\boldsymbol{p}_1)\right)^{n_1} \ \left(a^\dagger(\boldsymbol{p}_2)\right)^{n_2} |0\rangle + n_2\delta^3(\boldsymbol{k}-\boldsymbol{p}_2) \ \left(a^\dagger(\boldsymbol{p}_1)\right)^{n_1} \ \left(a^\dagger(\boldsymbol{p}_2)\right)^{n_2} |0\rangle \\
&= \left[n_1\delta^3(\boldsymbol{k}-\boldsymbol{p}_1) + n_2\delta^3(\boldsymbol{k}-\boldsymbol{p}_2)\right] \ \left(a^\dagger(\boldsymbol{p}_1)\right)^{n_1} \ \left(a^\dagger(\boldsymbol{p}_2)\right)^{n_2} |0\rangle\,.
\end{aligned}$$

Using this result we get,

$$H \left(a^\dagger(\boldsymbol{p}_1)\right)^{n_1} \ \left(a^\dagger(\boldsymbol{p}_2)\right)^{n_2} |0\rangle = \left[E_0 + n_1\omega_{\boldsymbol{p_1}} + n_2\omega_{\boldsymbol{p_2}}\right] \left(a^\dagger(\boldsymbol{p}_1)\right)^{n_1} \ \left(a^\dagger(\boldsymbol{p}_2)\right)^{n_2} |0\rangle\,,$$

and,

$$\boldsymbol{P} \left(a^\dagger(\boldsymbol{p}_1)\right)^{n_1} \ \left(a^\dagger(\boldsymbol{p}_2)\right)^{n_2} |0\rangle = \left[n_1\boldsymbol{p}_1 + n_2\boldsymbol{p}_2\right] \left(a^\dagger(\boldsymbol{p}_1)\right)^{n_1} \ \left(a^\dagger(\boldsymbol{p}_2)\right)^{n_2} |0\rangle\,.$$

Thus this is a state with the energy, $n_1\omega_{\boldsymbol{p_1}} + n_2\omega_{\boldsymbol{p_2}}$ and momentum $n_1\boldsymbol{p_1} + n_2\boldsymbol{p_2}$ above the vacuum. Since the vacuum is interpreted to be a zero particle/ no particle state, this state can be interpreted as a state of n_1 noninteracting particles of mass m and linear 3-momentum $\boldsymbol{p}_1$ and n_2 noninteracting particles of mass m and linear momentum $\boldsymbol{p}_2$.

3. In the lectures I claimed the two point function (Wightman function) determines all higher order correlation functions in theory of the free real quantum scalar field. Verify this claim by proving this the following for the case of the 4-point correlation function, namely

$$\left\langle 0 \middle| \varphi(x)\, \varphi(y)\, \varphi(z)\, \varphi(w) \middle| 0 \right\rangle = \Delta_+(x,y)\, \Delta_+(z,w) + \Delta_+(x,z)\, \Delta_+(y,w) + \Delta_+(x,w)\, \Delta_+(y,z)$$

(10 points)

Solution: To start with we need the expression,

$$\varphi(w)|0\rangle = \int \frac{d^3\boldsymbol{p}}{\sqrt{(2\pi)^3\, 2\omega_{\boldsymbol{p}}}}\ e^{ip.w}\Big|_{p^0=\omega_{\boldsymbol{p}}} a^\dagger(\boldsymbol{p})|0\rangle$$

and then we get

$$
\begin{aligned}
\varphi(z)\,\varphi(w)\big|0\rangle &= \int \frac{d^3\boldsymbol{q}}{\sqrt{(2\pi)^3\,2\omega_{\boldsymbol{q}}}}\frac{d^3\boldsymbol{p}}{\sqrt{(2\pi)^3\,2\omega_{\boldsymbol{p}}}}\; e^{iq.z}\big|_{q^0=\omega_{\boldsymbol{q}}}\; e^{ip.w}\big|_{p^0=\omega_{\boldsymbol{p}}}\; a^\dagger(\boldsymbol{q})\,a^\dagger(\boldsymbol{p})\big|0\rangle + \int \frac{d^3\boldsymbol{q}}{\sqrt{(2\pi)^3\,2\omega_{\boldsymbol{q}}}}\frac{d^3\boldsymbol{p}}{\sqrt{(2\pi)^3\,2\omega_{\boldsymbol{p}}}}\; e^{-iq.z}\big|_{q^0=\omega_{\boldsymbol{q}}}\; e^{ip.w}\big|_{p^0=\omega_{\boldsymbol{p}}}\; a(\\
&= \int \frac{d^3\boldsymbol{q}}{\sqrt{(2\pi)^3\,2\omega_{\boldsymbol{q}}}}\frac{d^3\boldsymbol{p}}{\sqrt{(2\pi)^3\,2\omega_{\boldsymbol{p}}}}\; e^{iq.z}\big|_{q^0=\omega_{\boldsymbol{q}}}\; e^{ip.w}\big|_{p^0=\omega_{\boldsymbol{p}}}\; a^\dagger(\boldsymbol{q})\,a^\dagger(\boldsymbol{p})\big|0\rangle + \int \frac{d^3\boldsymbol{q}}{\sqrt{(2\pi)^3\,2\omega_{\boldsymbol{q}}}}\frac{d^3\boldsymbol{p}}{\sqrt{(2\pi)^3\,2\omega_{\boldsymbol{p}}}}\; e^{-iq.z}\big|_{q^0=\omega_{\boldsymbol{q}}}\; e^{ip.w}\big|_{p^0=\omega_{\boldsymbol{p}}}\; \Big[\\
&= \int \frac{d^3\boldsymbol{q}}{\sqrt{(2\pi)^3\,2\omega_{\boldsymbol{q}}}}\frac{d^3\boldsymbol{p}}{\sqrt{(2\pi)^3\,2\omega_{\boldsymbol{p}}}}\; e^{iq.z}\big|_{q^0=\omega_{\boldsymbol{q}}}\; e^{ip.w}\big|_{p^0=\omega_{\boldsymbol{p}}}\; a^\dagger(\boldsymbol{q})\,a^\dagger(\boldsymbol{p})\big|0\rangle + \int \frac{d^3\boldsymbol{p}}{(2\pi)^3\,2\omega_{\boldsymbol{p}}}\; e^{-ip.(z-w)}\Big|_{p^0=\omega_{\boldsymbol{p}}}\;\big|0\rangle\\
&= \int \frac{d^3\boldsymbol{q}}{\sqrt{(2\pi)^3\,2\omega_{\boldsymbol{q}}}}\frac{d^3\boldsymbol{p}}{\sqrt{(2\pi)^3\,2\omega_{\boldsymbol{p}}}}\; e^{iq.z}\big|_{q^0=\omega_{\boldsymbol{q}}}\; e^{ip.w}\big|_{p^0=\omega_{\boldsymbol{p}}}\; a^\dagger(\boldsymbol{q})\,a^\dagger(\boldsymbol{p})\big|0\rangle + \big|0\rangle\Delta_+(z,y).
\end{aligned}
$$

Similarly taking a hermitian conjugate,

$$
\langle 0\big|\varphi(x)\,\varphi(y) = \big(\varphi(y)\,\varphi(x)\big|0\rangle\big)^\dagger = \Delta_-(y,x)\langle 0\big| + \int \frac{d^3\boldsymbol{s}}{\sqrt{(2\pi)^3\,2\omega_{\boldsymbol{s}}}}\frac{d^3\boldsymbol{r}}{\sqrt{(2\pi)^3\,2\omega_{\boldsymbol{r}}}}\; e^{-is.x}\big|_{s^0=\omega_{\boldsymbol{s}}}\; e^{-ir.y}\big|_{r^0=\omega_{\boldsymbol{p}}}\;\langle 0\big|a(\boldsymbol{s})\,a(\boldsymbol{r}).
$$

Thus, $G(x,y,w,z)\equiv\langle 0\big|\varphi(x)\,\varphi(y)\,\varphi(z)\,\varphi(w)\big|0\rangle$ is given by the overlap,

$$
\begin{aligned}
G(x,y,w,z) &= \Delta_-(y,x)\int \frac{d^3\boldsymbol{q}}{\sqrt{(2\pi)^3\,2\omega_{\boldsymbol{q}}}}\frac{d^3\boldsymbol{p}}{\sqrt{(2\pi)^3\,2\omega_{\boldsymbol{p}}}}\; e^{iq.z}\big|_{q^0=\omega_{\boldsymbol{q}}}\; e^{ip.w}\big|_{p^0=\omega_{\boldsymbol{p}}}\; \overset{0}{\cancel{\langle 0\big|a^\dagger(\boldsymbol{q})\,a^\dagger(\boldsymbol{p})\big|0\rangle}} + \Delta_-(y,x)\,\underbrace{\langle 0\big|0\rangle}_{=1}\,\Delta_+(z,y)\\
&\quad + \int \frac{d^3\boldsymbol{s}}{\sqrt{(2\pi)^3\,2\omega_{\boldsymbol{s}}}}\frac{d^3\boldsymbol{r}}{\sqrt{(2\pi)^3\,2\omega_{\boldsymbol{r}}}}\frac{d^3\boldsymbol{q}}{\sqrt{(2\pi)^3\,2\omega_{\boldsymbol{q}}}}\frac{d^3\boldsymbol{p}}{\sqrt{(2\pi)^3\,2\omega_{\boldsymbol{p}}}}\; e^{-is.x}\big|_{s^0=\omega_{\boldsymbol{s}}}\; e^{-ir.y}\big|_{r^0=\omega_{\boldsymbol{r}}}\; e^{iq.z}\big|_{q^0=\omega_{\boldsymbol{q}}}\; e^{ip.w}\big|_{p^0=\omega_{\boldsymbol{p}}}\;\langle 0\big|a(\boldsymbol{s})\,a(\boldsymbol{r})a^\dagger(\boldsymbol{q})\,a^\dagger\\
&\qquad + \int \frac{d^3\boldsymbol{s}}{\sqrt{(2\pi)^3\,2\omega_{\boldsymbol{s}}}}\frac{d^3\boldsymbol{r}}{\sqrt{(2\pi)^3\,2\omega_{\boldsymbol{r}}}}\; e^{-is.x}\big|_{s^0=\omega_{\boldsymbol{s}}}\; e^{-ir.y}\big|_{r^0=\omega_{\boldsymbol{r}}}\; \overset{0}{\cancel{\langle 0\big|a(\boldsymbol{s})\,a(\boldsymbol{r})\big|0\rangle}}\Delta_+(z,y)\\
&= \Delta_-(y,x)\;\Delta_+(z,w)\\
&\quad + \int \frac{d^3\boldsymbol{s}}{\sqrt{(2\pi)^3\,2\omega_{\boldsymbol{s}}}}\frac{d^3\boldsymbol{r}}{\sqrt{(2\pi)^3\,2\omega_{\boldsymbol{r}}}}\frac{d^3\boldsymbol{q}}{\sqrt{(2\pi)^3\,2\omega_{\boldsymbol{q}}}}\frac{d^3\boldsymbol{p}}{\sqrt{(2\pi)^3\,2\omega_{\boldsymbol{p}}}}\; e^{-is.x}\big|_{s^0=\omega_{\boldsymbol{s}}}\; e^{-ir.y}\big|_{r^0=\omega_{\boldsymbol{r}}}\; e^{iq.z}\big|_{q^0=\omega_{\boldsymbol{q}}}\; e^{ip.w}\big|_{p^0=\omega_{\boldsymbol{p}}}\;\langle 0\big|a(\boldsymbol{s})\,a(\boldsymbol{r})a^\dagger(\boldsymbol{q})\,a^\dagger
\end{aligned}
\tag{3}
$$

Now, we simplify,

$$
\begin{aligned}
\langle 0\big|a(\boldsymbol{s})\,a(\boldsymbol{r})a^\dagger(\boldsymbol{q})\,a^\dagger(\boldsymbol{p})\big|0\rangle &= \langle 0\big|a(\boldsymbol{s})\,\Big[a(\boldsymbol{r}),a^\dagger(\boldsymbol{q})\Big]\;a^\dagger(\boldsymbol{p})\big|0\rangle + \langle 0\big|a(\boldsymbol{s})a^\dagger(\boldsymbol{q})\,a(\boldsymbol{r})\,a^\dagger(\boldsymbol{p})\big|0\rangle\\
&= \delta^3(\boldsymbol{r}-\boldsymbol{q})\langle 0\big|a(\boldsymbol{s})\,a^\dagger(\boldsymbol{p})\big|0\rangle + \langle 0\big|a(\boldsymbol{s})a^\dagger(\boldsymbol{q})\,a(\boldsymbol{r})\,a^\dagger(\boldsymbol{p})\big|0\rangle\\
&= \delta^3(\boldsymbol{r}-\boldsymbol{q})\langle 0\big|\,\Big[a(\boldsymbol{s}),a^\dagger(\boldsymbol{p}\Big])\big|0\rangle + \langle 0\big|\,\Big[a(\boldsymbol{s}),a^\dagger(\boldsymbol{q})\Big]\;\Big[a(\boldsymbol{r}),a^\dagger(\boldsymbol{p})\Big]\,\big|0\rangle\\
&= \delta^3(\boldsymbol{r}-\boldsymbol{q})\;\delta^3(\boldsymbol{s}-\boldsymbol{p}) + \delta^3(\boldsymbol{s}-\boldsymbol{q})\;\delta^3(\boldsymbol{r}-\boldsymbol{p}).
\end{aligned}
$$

Using this result back in (3), we get

$$
\begin{aligned}
G(x,y,w,z) &= \Delta_-(y,x)\;\Delta_+(z,w)\\
&\quad + \int \frac{d^3\boldsymbol{s}}{\sqrt{(2\pi)^3\,2\omega_{\boldsymbol{s}}}}\frac{d^3\boldsymbol{r}}{\sqrt{(2\pi)^3\,2\omega_{\boldsymbol{r}}}}\frac{d^3\boldsymbol{q}}{\sqrt{(2\pi)^3\,2\omega_{\boldsymbol{q}}}}\frac{d^3\boldsymbol{p}}{\sqrt{(2\pi)^3\,2\omega_{\boldsymbol{p}}}}\; e^{-is.x}\big|_{s^0=\omega_{\boldsymbol{s}}}\; e^{-ir.y}\big|_{r^0=\omega_{\boldsymbol{r}}}\; e^{iq.z}\big|_{q^0=\omega_{\boldsymbol{q}}}\; e^{ip.w}\big|_{p^0=\omega_{\boldsymbol{p}}}\;\delta^3(\boldsymbol{r}-\boldsymbol{q})\;\delta^3(\boldsymbol{s}-\boldsymbol{p})\\
&\quad + \int \frac{d^3\boldsymbol{s}}{\sqrt{(2\pi)^3\,2\omega_{\boldsymbol{s}}}}\frac{d^3\boldsymbol{r}}{\sqrt{(2\pi)^3\,2\omega_{\boldsymbol{r}}}}\frac{d^3\boldsymbol{q}}{\sqrt{(2\pi)^3\,2\omega_{\boldsymbol{q}}}}\frac{d^3\boldsymbol{p}}{\sqrt{(2\pi)^3\,2\omega_{\boldsymbol{p}}}}\; e^{-is.x}\big|_{s^0=\omega_{\boldsymbol{s}}}\; e^{-ir.y}\big|_{r^0=\omega_{\boldsymbol{r}}}\; e^{iq.z}\big|_{q^0=\omega_{\boldsymbol{q}}}\; e^{ip.w}\big|_{p^0=\omega_{\boldsymbol{p}}}\;\delta^3(\boldsymbol{s}-\boldsymbol{q})\;\delta^3(\boldsymbol{r}-\boldsymbol{p}).\\
&= \underbrace{\Delta_-(y,x)}_{\Delta_+(x,y)}\;\Delta_+(z,w)\\
&\quad + \left(\frac{d^3\boldsymbol{p}}{(2\pi)^3\,2\omega_{\boldsymbol{p}}}\; e^{-ip.(x-w)}\Big|_{p^0=\omega_{\boldsymbol{p}}}\right)\left(\int \frac{d^3\boldsymbol{q}}{(2\pi)^3\,2\omega_{\boldsymbol{q}}}\; e^{-iq.(y-.z)}\Big|_{q^0=\omega_{\boldsymbol{q}}}\right)\\
&\qquad + \left(\frac{d^3\boldsymbol{q}}{(2\pi)^3\,2\omega_{\boldsymbol{q}}}\; e^{-iq.(x-z)}\Big|_{q^0=\omega_{\boldsymbol{q}}}\right)\left(\frac{d^3\boldsymbol{p}}{(2\pi)^3\,2\omega_{\boldsymbol{p}}}\; e^{-ip.(y-w)}\Big|_{p^0=\omega_{\boldsymbol{p}}}\right)\\
&= \Delta_+(x,y)\;\Delta_+(z,w) + \Delta_+(x,w)\;\Delta_+(y,z) + \Delta_+(x,z)\;\Delta_+(y,w).
\end{aligned}
$$

4. Consider $n$free point particles of masses $m_1, m_2, \ldots, m_n$. Let p be the total energy-momentum 4-vector, $p = (p_1 + p_2 + \ldots + p_n)$. Show that,

$$p^2 \geq (m_1 + m_2 + \ldots + m_n)^2 .$$

(4 points)

Solution: The Let $\left(E_i^{CM}, \boldsymbol{p}_i^{CM}\right)$ represent the CM frame 4-momentum of the i-the particle of mass m_i. Here

$$E_i^{CM} = \sqrt{m_i^2 + \left(\boldsymbol{p}_i^{CM}\right)^2}$$

Then the total 4-momentum in the CM frame is,

$$P^{CM} = \left(\sum_{i=1}^{n} E_i^{CM}, 0\right)$$

and hence the invariant norm squared of the total 4-momentum is,

$$\begin{aligned} p^2 = \left(p^{CM}\right)^2 &= \left(\sum_{i=1}^{n} E_i^{CM}\right)^2 \\ &= \left(\sum_{i=1}^{n} \sqrt{m_i^2 + \left(\boldsymbol{p}_i^{CM}\right)^2}\right)^2 \\ &\geq \left(\sum_{i=1}^{n} m_i\right)^2 , \end{aligned}$$

since $\left(\boldsymbol{p}_i^{CM}\right)^2 \geq 0$.

5. Show that the quantum particles obtained by quantizing the free real scalar field obey Bose statistics, i.e. the multi-particle wave functions or a ket/vector representing multi-particle states are symmetric under particle exchange.

(2 points)

Solution: Consider a n-particle state of the quantum scalar field, given by

$$\left|\psi(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)\right\rangle = \varphi(\boldsymbol{x}_1, t) \ldots \varphi(\boldsymbol{x}_n, t)\left|0\right\rangle$$

which represents n particles, each of mass m localized at positions $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ at some given fixed time t. If we swap the i-th located at $\boldsymbol{x}_i$ and the j-th particle located at $\boldsymbol{x}_j$, i.e starting from the state

$$\left|\psi(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_i, \ldots, \boldsymbol{x}_j, \ldots, \boldsymbol{x}_n)\right\rangle = \varphi(\boldsymbol{x}_1, t) \ldots \varphi(\boldsymbol{x}_i, t) \ldots \varphi(\boldsymbol{x}_j, t) \ldots \varphi(\boldsymbol{x}_n, t)\left|0\right\rangle$$

we create the state,

$$\left|\psi(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_j, \ldots, \boldsymbol{x}_i, \ldots, \boldsymbol{x}_n)\right\rangle = \varphi(\boldsymbol{x}_1, t) \ldots \varphi(\boldsymbol{x}_j, t) \ldots \varphi(\boldsymbol{x}_i, t) \ldots \varphi(\boldsymbol{x}_n, t)\left|0\right\rangle$$

But since at equal time t, the $\varphi(\boldsymbol{x}, t)$'s for different $\boldsymbol{x}$'s commute, we can change their order using the equal time commutation rule! In particular in this case we can restore the order of the i-th and j-th particle. This proves that

$$\left|\psi(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_i, \ldots, \boldsymbol{x}_j, \ldots, \boldsymbol{x}_n)\right\rangle = \left|\psi(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_j, \ldots, \boldsymbol{x}_i, \ldots, \boldsymbol{x}_n)\right\rangle,$$

i.e. the state is *symmetric* under the swap of the i-th and j-th particle. So the quanta of excitation of scalar quantum fields obey *Bose statistics* (bosons).

6. In class I mentioned that boost symmetry does not lead to any new Noether charge(s) (unlike translation symmetry and rotational symmetry which lead to energy-momentum 4-vector and angular momentum 3-vector as conserved Noether charges). Prove this.

(2 points)

Solution: The boost charge via the Noether process is,

$$J^{0i} \equiv \int d^3\boldsymbol{x} \, \left(x^0 \, T^{0i} - x^i \, T^{00}\right),$$

where $T^{\mu\nu}$ is the Belinfante-Rosenfield symmetric stress tensor. Next by the definition of the CM,

$$\int d^3\boldsymbol{x} \, x^i \, T^{00} = X^i_{CM} \, E.$$

and,

$$\int d^3\boldsymbol{x} \, x^0 \, T^{0i} = x^0 \int d^3\boldsymbol{x} \, T^{0i} = x^0 \, P^i,$$

where P^i, E and the i-th component of relativistic linear momentum and relativistic energy (zeroth component) of the total linear 4-momentum of the system. Thus the boost charge is,

$$J^{0i} = x^0 \, P^i - X^i_{CM} \, E$$

Since for an isolated system the CM moves in a straight line,

$$X^i_{CM} = \beta^i \, x^0 + X^i_0 = \frac{P^i}{E} x^0 + X^i_0$$

where X^i_0 is a constant of integration (initial position of the CM). In particular this constant can be set to zero using translational symmetry, $X^i_0 = 0$. This gives,

$$J^{0i} = x^0 P^i - X^i_{CM} \, E = 0.$$

Thus boost symmetry does not lead to any new conserved charge.

7. Consider the theory of the free complex scalar field specified by the lagrangian density,

$$\mathcal{L} = (\partial^\mu \Phi)^\dagger \, (\partial_\mu \Phi) - m^2 \Phi^\dagger \, \Phi.$$

A. Write down the equal time canonical commutation relations.

B. Work out the Hamiltonian and Linear momentum. (Hint: These can both be derived from the canonical stress tensor)

C. Write down the solution for the field operator Φ as a mode sum. (Hint: Resolve Φ into real and imaginary parts $\Phi = \frac{\varphi_1 + i\varphi_2}{\sqrt{2}}$ and then substitute the mode expansion for the real scalar field)

D. What is the algebra obeyed by the mode coefficient operators of the positive energy modes (say $b(\boldsymbol{k})$) and negative energy modes (say $d^\dagger(\boldsymbol{k}')$) i.e. $\left[b, b^\dagger\right]$, $\left[d, d^\dagger\right]$, $\left[b, d^\dagger\right]$, $\left[d, b^\dagger\right]$

E. Write down the Hamiltonian and Linear momentum operator as momentum space mode sum.

F. Particle interpretation: Show that the Hilbert space of states of the free quantum complex scalar field is spanned by states of *two* kinds of noninteracting particles of mass m. (Hint:

Consider the two singly excited states, $b^\dagger(\boldsymbol{k})\,|0\rangle$, $d^\dagger(\boldsymbol{k}')\,|0\rangle$)

G. These two particles (b-particles and d-particles) have identical masses m and identical spin (zero). So what distinguishes these two types of particles?

(3 + 3 + 2 + 4 + 4 + 4 + 2 = 22 points)

Solution: A. The conjugate momentum to Φ is,

$$\Pi = \frac{\partial \mathcal{L}}{\partial \dot{\Phi}} = \dot{\Phi}^\dagger,$$

and the conjugate momentum of $\Phi^\dagger$ is,

$$\Pi^\dagger = \frac{\partial \mathcal{L}}{\partial \Phi^\dagger} = \dot{\Phi}.$$

So the ETCCR are,

$$[\Phi(x), \Pi(y)]_{x^0=y^0} = \left[\Phi^\dagger(x), \Pi^\dagger(y)\right]_{x^0=y^0} = i\delta^3(\boldsymbol{x}-\boldsymbol{y})$$

or,

$$\left[\Phi(x), \dot{\Phi}^\dagger(y)\right]_{x^0=y^0} = \left[\Phi^\dagger(x), \dot{\Phi}(y)\right]_{x^0=y^0} = i\delta^3(\boldsymbol{x}-\boldsymbol{y}),$$

and,

$$[\Phi(x), \Phi(y)]_{x^0=y^0} = \left[\Phi^\dagger(x), \Phi^\dagger(y)\right]_{x^0=y^0} = \left[\Phi(x), \Phi^\dagger(y)\right]_{x^0=y^0} = 0,$$

and

$$[\Pi(x), \Pi(y)]_{x^0=y^0} = \left[\Pi^\dagger(x), \Pi^\dagger(y)\right]_{x^0=y^0} = \left[\Pi(x), \Pi^\dagger(y)\right]_{x^0=y^0} = 0,$$

or,

$$\left[\dot{\Phi}(x), \dot{\Phi}(y)\right]_{x^0=y^0} = \left[\dot{\Phi}^\dagger(x), \dot{\Phi}^\dagger(y)\right]_{x^0=y^0} = \left[\dot{\Phi}(x), \dot{\Phi}^\dagger(y)\right]_{x^0=y^0} = 0.$$

B. The canonical stress tensor for the complex scalar is,

$$\begin{aligned}\Theta^{\mu\nu} &= \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \Phi\right)}\partial^\nu \Phi + \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \Phi^\dagger\right)}\partial^\nu \Phi^\dagger - \eta^{\mu\nu}\mathcal{L} \\ &= \left(\partial^\mu \Phi\right)^\dagger \partial^\nu \Phi + \partial^\mu \Phi\, \partial^\nu \Phi^\dagger - \eta^{\mu\nu}\left(\left(\partial^\lambda \Phi\right)^\dagger \left(\partial_\lambda \Phi\right) - m^2 \Phi^\dagger\, \Phi\right).\end{aligned}$$

Then the Hamiltonian density is,

$$\begin{aligned}\Theta^{00} &= \dot{\Phi}^\dagger \dot{\Phi} + \dot{\Phi}\, \dot{\Phi}^\dagger - \left(\left(\partial^\lambda \Phi\right)^\dagger \left(\partial_\lambda \Phi\right) - m^2 \Phi^\dagger\, \Phi\right) \\ &= 2\left|\dot{\Phi}\right|^2 - \left(\left|\dot{\Phi}\right|^2 - \left|\boldsymbol{\nabla}\Phi\right|^2 - m^2\left|\Phi\right|^2\right) \\ &= \left|\dot{\Phi}\right|^2 + \left|\boldsymbol{\nabla}\Phi\right|^2 + m^2\left|\Phi\right|^2.\end{aligned}$$

and the Hamiltonian is,

$$H = \int d^3\boldsymbol{x}\, \Theta^{00} = \int d^3\boldsymbol{x}\, \left(\left|\dot{\Phi}\right|^2 + \left|\boldsymbol{\nabla}\Phi\right|^2 + m^2\left|\Phi\right|^2\right).$$

Similarly the momentum density is,

$$\Theta^{0i} = \dot{\Phi}^\dagger \partial^i \Phi + \dot{\Phi}\, \partial^i \Phi^\dagger - \cancelto{0}{\eta^{0i}} \left(\left(\partial^\lambda \Phi\right)^\dagger (\partial_\lambda \Phi) - m^2 \Phi^\dagger\, \Phi \right)$$
$$= - \left(\dot{\Phi}^\dagger \partial_i \Phi + \dot{\Phi}\, \partial_i \Phi^\dagger \right).$$

and the momentum is,

$$\boldsymbol{P} = - \int d^3\boldsymbol{x}\, \left(\dot{\Phi}^\dagger \boldsymbol{\nabla} \Phi + \dot{\Phi}\, \nabla \Phi^\dagger \right).$$

C. Resolving Φ into real and imaginary parts

$$\Phi = \frac{\varphi_1 + i\varphi_2}{\sqrt{2}}$$

we see that the complex scalar lagrangian splits into two noninteracting free real scalar field Lagrangians

$$\mathcal{L} = \frac{1}{2} (\partial \varphi_1)^2 - \frac{m^2}{2} \varphi_1^2 + \frac{1}{2} (\partial \varphi_2)^2 - \frac{m^2}{2} \varphi_2^2$$

The form of the solution to the two resultant free scalar field operators is,

$$\varphi_1(x) = \int \frac{d^3\boldsymbol{k}}{\sqrt{(2\pi)^3\, 2\omega_{\boldsymbol{k}}}} \left(a_1(\boldsymbol{k})\, e^{-ik.x} + a_1^\dagger(\boldsymbol{k})\, e^{ik.x} \right)_{k^0 = \omega_{\boldsymbol{k}}},$$

$$\varphi_2(x) = \int \frac{d^3\boldsymbol{k}}{\sqrt{(2\pi)^3\, 2\omega_{\boldsymbol{k}}}} \left(a_2(\boldsymbol{k})\, e^{-ik.x} + a_2^\dagger(\boldsymbol{k})\, e^{ik.x} \right)_{k^0 = \omega_{\boldsymbol{k}}}.$$

Then the complex scalar field is given by the mode sum,

$$\Phi(x) = \int \frac{d^3\boldsymbol{k}}{\sqrt{(2\pi)^3\, 2\omega_{\boldsymbol{k}}}} \left(b(\boldsymbol{k})\, e^{-ik.x} + d^\dagger(\boldsymbol{k})\, e^{ik.x} \right)_{k^0 = \omega_{\boldsymbol{k}}}$$

where we have defined,

$$b(\boldsymbol{k}) \equiv \frac{a_1(\boldsymbol{k}) + i\, a_2(\boldsymbol{k})}{\sqrt{2}},$$
$$d^\dagger(\boldsymbol{k}) \equiv \frac{a_1^\dagger(\boldsymbol{k}) + i\, a_2^\dagger(\boldsymbol{k})}{\sqrt{2}}.$$

D. We start with the algebra of mode coefficients for φ_1, φ_2, namely,

$$\left[a_1(\boldsymbol{k}), a_1^\dagger(\boldsymbol{k}')\right] = \left[a_2(\boldsymbol{k}), a_2^\dagger(\boldsymbol{k}')\right] = \delta^3(\boldsymbol{k} - \boldsymbol{k}'),$$

$$\left[a_1(\boldsymbol{k}), a_1(\boldsymbol{k}')\right] = \left[a_2(\boldsymbol{k}), a_2(\boldsymbol{k}')\right] = \left[a_1^\dagger(\boldsymbol{k}), a_1^\dagger(\boldsymbol{k}')\right] = \left[a_2^\dagger(\boldsymbol{k}), a_2^\dagger(\boldsymbol{k}')\right] = 0,$$
$$\left[a_1(\boldsymbol{k}), a_2(\boldsymbol{k}')\right] = \left[a_1(\boldsymbol{k}), a_2^\dagger(\boldsymbol{k}')\right] = \left[a_1^\dagger(\boldsymbol{k}), a_2(\boldsymbol{k}')\right] = \left[a_1^\dagger(\boldsymbol{k}), a_2^\dagger(\boldsymbol{k}')\right] = 0.$$

Using these we determine,

$$\left[b(\boldsymbol{k}), b^\dagger(\boldsymbol{k}')\right] = \left[\frac{a_1(\boldsymbol{k}) + i\, a_2(\boldsymbol{k})}{\sqrt{2}}, \frac{a_1^\dagger(\boldsymbol{k}') - i\, a_2^\dagger(\boldsymbol{k}')}{\sqrt{2}}\right] = \frac{1}{2}\left[a_1(\boldsymbol{k}), a_1^\dagger(\boldsymbol{k}')\right] + \frac{1}{2}\left[a_2(\boldsymbol{k}), a_2^\dagger(\boldsymbol{k}')\right] = \delta^3(\boldsymbol{k} - \boldsymbol{k}'),$$

$$\left[d(\boldsymbol{k}), d^\dagger(\boldsymbol{k}')\right] = \left[\frac{a_1(\boldsymbol{k}) - i\, a_2(\boldsymbol{k})}{\sqrt{2}}, \frac{a_1^\dagger(\boldsymbol{k}') + i\, a_2^\dagger(\boldsymbol{k}')}{\sqrt{2}}\right] = \frac{1}{2}\left[a_1(\boldsymbol{k}), a_1^\dagger(\boldsymbol{k}')\right] + \frac{1}{2}\left[a_2(\boldsymbol{k}), a_2^\dagger(\boldsymbol{k}')\right] = \delta^3(\boldsymbol{k}-\boldsymbol{k}'),$$

$$\left[b(\boldsymbol{k}), d^\dagger(\boldsymbol{k}')\right] = \left[\frac{a_1(\boldsymbol{k}) + i\, a_2(\boldsymbol{k})}{\sqrt{2}}, \frac{a_1^\dagger(\boldsymbol{k}') + i\, a_2^\dagger(\boldsymbol{k}')}{\sqrt{2}}\right] = \frac{1}{2}\left[a_1(\boldsymbol{k}), a_1^\dagger(\boldsymbol{k}')\right] - \frac{1}{2}\left[a_2(\boldsymbol{k}), a_2^\dagger(\boldsymbol{k}')\right] = 0,$$

$$\left[b^\dagger(\boldsymbol{k}), d(\boldsymbol{k}')\right] = \left[\frac{a_1(\boldsymbol{k}) - i\, a_2(\boldsymbol{k})}{\sqrt{2}}, \frac{a_1^\dagger(\boldsymbol{k}') - i\, a_2^\dagger(\boldsymbol{k}')}{\sqrt{2}}\right] = \frac{1}{2}\left[a_1(\boldsymbol{k}), a_1^\dagger(\boldsymbol{k}')\right] - \frac{1}{2}\left[a_2(\boldsymbol{k}), a_2^\dagger(\boldsymbol{k}')\right] = 0,$$

$$\left[b(\boldsymbol{k}), b(\boldsymbol{k}')\right] = \left[\frac{a_1(\boldsymbol{k}) + i\, a_2(\boldsymbol{k})}{\sqrt{2}}, \frac{a_1(\boldsymbol{k}') + i\, a_2(\boldsymbol{k}')}{\sqrt{2}}\right] = 0,$$

$$\left[d(\boldsymbol{k}), d(\boldsymbol{k}')\right] = \left[\frac{a_1(\boldsymbol{k}) - i\, a_2(\boldsymbol{k})}{\sqrt{2}}, \frac{a_1(\boldsymbol{k}') - i\, a_2(\boldsymbol{k}')}{\sqrt{2}}\right] = 0.$$

E.Hamiltonian and Linear momentum operator as momentum space mode sum:

$$\begin{aligned} H &= H[\varphi_1] + H[\varphi_2] \\ &= \int d^3\boldsymbol{k}\, \omega_{\boldsymbol{k}} \left(a_1(\boldsymbol{k}) a_1^\dagger(\boldsymbol{k}) + a_1^\dagger(\boldsymbol{k}) a_1(\boldsymbol{k})\right) + \int d^3\boldsymbol{k}\, \omega_{\boldsymbol{k}} \left(a_2(\boldsymbol{k}) a_2^\dagger(\boldsymbol{k}) + a_2^\dagger(\boldsymbol{k}) a_2(\boldsymbol{k})\right), \end{aligned}$$

while,

$$\begin{aligned} \boldsymbol{P} &= \boldsymbol{P}[\varphi_1] + \boldsymbol{P}[\varphi_2] \\ &= \int d^3\boldsymbol{k}\, \boldsymbol{k} \left(a_1(\boldsymbol{k}) a_1^\dagger(\boldsymbol{k}) + a_1^\dagger(\boldsymbol{k}) a_1(\boldsymbol{k})\right) + \int d^3\boldsymbol{k}\, \boldsymbol{k} \left(a_2(\boldsymbol{k}) a_2^\dagger(\boldsymbol{k}) + a_2^\dagger(\boldsymbol{k}) a_2(\boldsymbol{k})\right). \end{aligned}$$

In terms of b, d modes:

$$\begin{aligned} a_1 a_1^\dagger + a_1^\dagger a_1 + a_2 a_2^\dagger + a_2^\dagger a_2 &= \frac{(b+d)\left(b^\dagger + d^\dagger\right)}{2} + \frac{\left(b^\dagger + d^\dagger\right)(b+d)}{2} - \frac{(b-d)\left(d^\dagger - b^\dagger\right)}{2} - \frac{\left(d^\dagger - b^\dagger\right)(b-d)}{2} \\ &= bb^\dagger + b^\dagger b + dd^\dagger + d^\dagger d \end{aligned}$$

Thus,

$$H = \int d^3\boldsymbol{k}\, \omega_{\boldsymbol{k}} \left(b(\boldsymbol{k})\, b^\dagger(\boldsymbol{k}) + b^\dagger(\boldsymbol{k})\, b(\boldsymbol{k})\right) + \int d^3\boldsymbol{k}\, \omega_{\boldsymbol{k}} \left(d(\boldsymbol{k})\, d^\dagger(\boldsymbol{k}) + d^\dagger(\boldsymbol{k})\, d(\boldsymbol{k})\right),$$

$$\boldsymbol{P} = \int d^3\boldsymbol{k}\, \boldsymbol{k} \left(b(\boldsymbol{k})\, b^\dagger(\boldsymbol{k}) + b^\dagger(\boldsymbol{k})\, b(\boldsymbol{k})\right) + \int d^3\boldsymbol{k}\, \boldsymbol{k} \left(d(\boldsymbol{k})\, d^\dagger(\boldsymbol{k}) + d^\dagger(\boldsymbol{k})\, d(\boldsymbol{k})\right).$$

F. Particle interpretation: Show that the Hilbert space of states of the free quantum complex scalar field is spanned by states of *two* kinds of noninteracting particles of mass m. (Hint: Consider the two singly excited states, $b^\dagger(\boldsymbol{k})\,|0\rangle$, $d^\dagger(\boldsymbol{k}')\,|0\rangle$). Define the vacuum to be state annihilated by b, d,

$$b(\boldsymbol{k})\,|0\rangle = d(\boldsymbol{k}')\,|0\rangle = 0.$$

Clearly this state has energy,

$$H\big|0\big\rangle = 2E_0\,\big|0\big\rangle$$

where E_0 is the divergent vacuum energy for a single free scalar field, and zero momentum:

$$\boldsymbol{P}\big|0\big\rangle = 0.$$

Then the "singly excited" state

$$b^\dagger(\boldsymbol{k})\,|0\rangle$$

have excitation energy $\omega_{\boldsymbol{k}}$

$$\begin{aligned}
(H - 2E_0)\; b^\dagger(\boldsymbol{k})\,|0\rangle &= \int d^3\boldsymbol{k}'\;\omega_{\boldsymbol{k}'}\; b^\dagger(\boldsymbol{k}')b(\boldsymbol{k}')b^\dagger(\boldsymbol{k})\,|0\rangle \\
&= \int d^3\boldsymbol{k}'\;\omega_{\boldsymbol{k}'}\; b^\dagger(\boldsymbol{k}')\left[b(\boldsymbol{k}'), b^\dagger(\boldsymbol{k})\right]|0\rangle \\
&= \omega_{\boldsymbol{k}}\; b^\dagger(\boldsymbol{k})\,|0\rangle
\end{aligned}$$

and 3-momentum $\boldsymbol{k}$

$$\begin{aligned}
\boldsymbol{P}\, b^\dagger(\boldsymbol{k})\,|0\rangle &= \int d^3\boldsymbol{k}'\;\boldsymbol{k}'\; b^\dagger(\boldsymbol{k}')b(\boldsymbol{k}')b^\dagger(\boldsymbol{k})\,|0\rangle \\
&= \int d^3\boldsymbol{k}'\;\boldsymbol{k}'\; b^\dagger(\boldsymbol{k}')\left[b(\boldsymbol{k}'), b^\dagger(\boldsymbol{k})\right]|0\rangle \\
&= \boldsymbol{k}\; b^\dagger(\boldsymbol{k})\,|0\rangle\,.
\end{aligned}$$

So this singly excited state represents a free spinless particle of mass m. Ditto for $d^\dagger(\boldsymbol{k}')\,|0\rangle$.

G. The distinction in the b and d type particles is in their $U(1)$ charge. The $U(1)$ charge is given by the expression,

$$\begin{aligned}
Q &=: i\int d^3\boldsymbol{x}\;\left(\Phi^\dagger\dot{\Phi} - \dot{\Phi}^\dagger\Phi\right) : \\
&= \int d^3\boldsymbol{k}\;\left(b^\dagger(\boldsymbol{k})\,b(\boldsymbol{k}) - d^\dagger(\boldsymbol{k})\,d(\boldsymbol{k})\right) \\
&= N_b - N_d.
\end{aligned}$$

Here N_b, N_d are the number operators counting the b-particles and d-particles respectively. Clearly a b-particle contributes $+1$ to the charge while a d-particle contributes a charge -1.