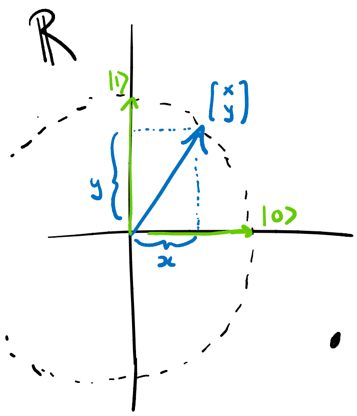
Lecture 11 — Linear Algebra of 1 Qubit

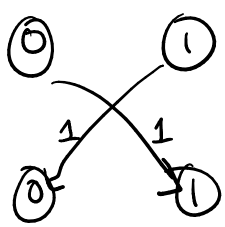
[[Say we have]] 1 qubit A, in state

[[x ampl. on "0", y ampl. on "1"]] $|v\rangle = x|0\rangle + y|1\rangle = \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$



[[What two instructions do we "have" for 1 qubit?]]

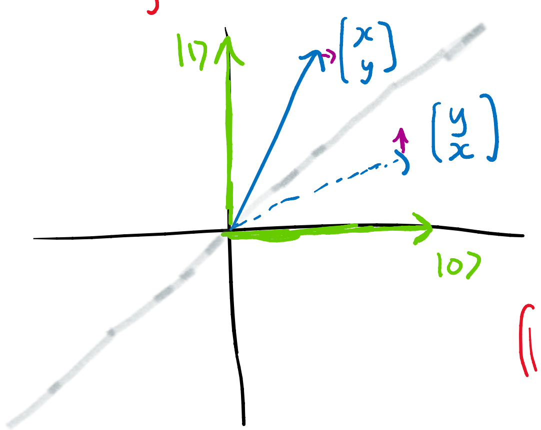
- Add 1 to A (real name: NOT on A)



$$\equiv \begin{array}{cc} & \text{input} \\ & \begin{array}{cc} |0\rangle & |1\rangle \end{array} \\ \text{output} \begin{array}{c} |0\rangle \\ |1\rangle \end{array} & \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{array} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$$

$$= y|0\rangle + x|1\rangle$$

[[To know where any $\begin{bmatrix} x \\ y \end{bmatrix}$ goes, enough to know where $|0\rangle$, $|1\rangle$ go.]]



[[Geometrically...]]

NOT = Reflection thru line at 45° $(= \pi/4)$

[[Checks: • Reflecting twice does nothing.
• Reflections are unitary (preserve unit-length).]]

- Hadamard on A

Official Quantum Name

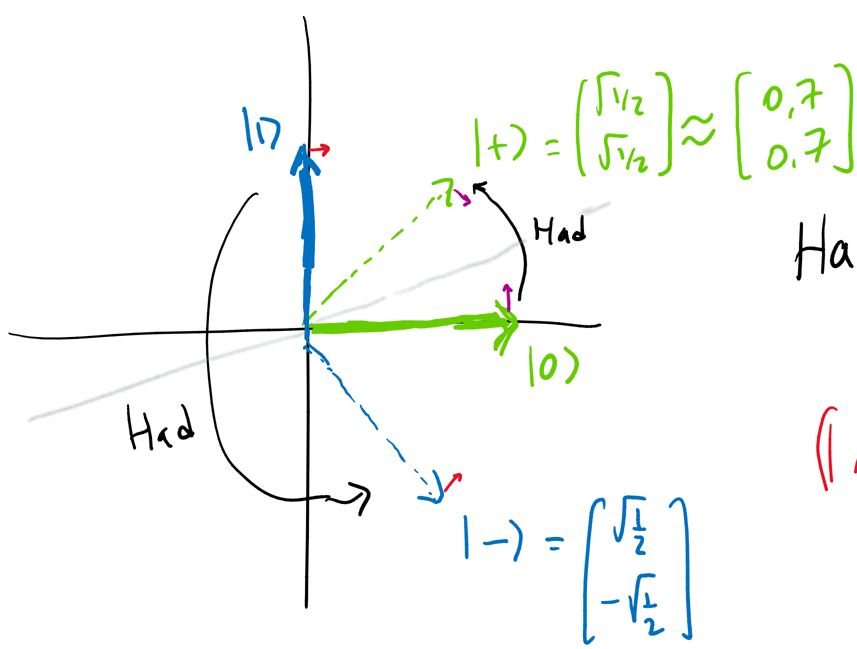
$|0\rangle \mapsto$ "ampl. $\sqrt{\frac{1}{2}}$ on $|0\rangle$, ampl. $\sqrt{\frac{1}{2}}$ on $|1\rangle$"

$$= \sqrt{\tfrac{1}{2}} \cdot |0\rangle + \sqrt{\tfrac{1}{2}}\, |1\rangle = \begin{bmatrix} \sqrt{\frac{1}{2}} \\ \sqrt{\frac{1}{2}} \end{bmatrix} = \text{“}|+\rangle\text{”}$$

$$|1\rangle \mapsto \sqrt{\tfrac{1}{2}} \cdot |0\rangle - \sqrt{\tfrac{1}{2}}\, |1\rangle = \begin{bmatrix} \sqrt{\frac{1}{2}} \\ -\sqrt{\frac{1}{2}} \end{bmatrix} = \text{“}|-\rangle\text{”}$$

[Bold move to name these vectors after ± symbols! But you'll come to like it!]

$$\text{Had} = \begin{array}{cc} & \text{in} \\ \text{out} & \begin{array}{c|cc} & |0\rangle & |1\rangle \\ \hline |0\rangle & +\sqrt{\frac{1}{2}} & +\sqrt{\frac{1}{2}} \\ |1\rangle & +\sqrt{\frac{1}{2}} & -\sqrt{\frac{1}{2}} \end{array} \end{array}$$



Had = Reflection thru line at angle 22.5° $= \pi/8$

[Again: self-inverse, preserves unit length]

[With 1 qubit, can still write some "code"...]

def R(A):
  NOT on A
  Had on A

[What does this do?]

Algebra:

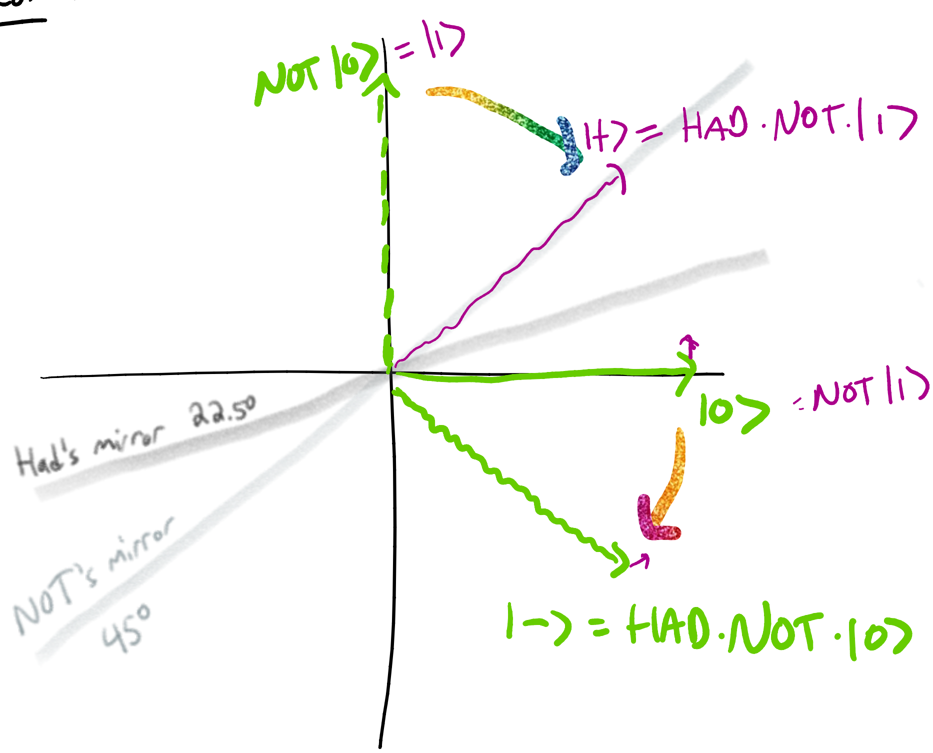
$|0\rangle \xrightarrow{\text{NOT}} |1\rangle \xrightarrow{\text{Had}} |-\rangle$

$|1\rangle \xrightarrow{\text{NOT}} |0\rangle \xrightarrow{\text{Had}} |+\rangle$

Matrix:

$$\left(\underbrace{\begin{bmatrix} \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} \\ \sqrt{\frac{1}{2}} & -\sqrt{\frac{1}{2}} \end{bmatrix}}_{\substack{\text{HAD} \\ \text{second}}} \cdot \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{\substack{\text{NOT} \\ \text{first}}} = \right) \quad \begin{array}{c c} & \begin{array}{cc} |0\rangle & \text{in} \quad |1\rangle \end{array} \\ \begin{array}{c} |0\rangle \\ \text{out} \\ |1\rangle \end{array} & \begin{bmatrix} \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} \\ -\sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} \end{bmatrix} \\ & \begin{array}{cc} \uparrow |-\rangle & \uparrow |+\rangle \end{array} \end{array}$$

Geom:



Overall behavior:
rotation by 45° clockwise!

Officially:
$R = \text{Rot}_{-45^\circ}$
because math "default" is counter clockwise!

[Note: rotations preserve unit length, but are <u>not</u> self-inverse!]

$$\left(Rot_{-45°}\right)^{-1} = \left(Rot_{-45°}\right)^{\dagger} = Rot_{45°}$$

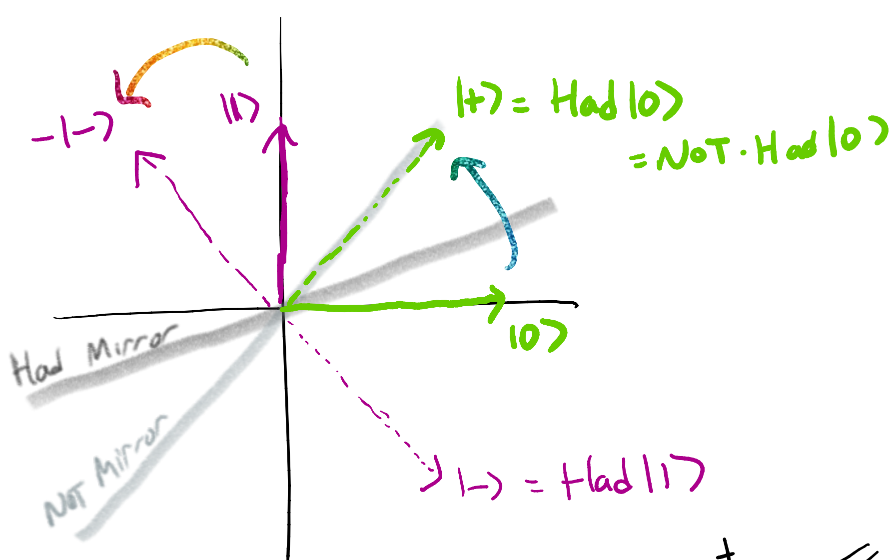
$$R^{-1} = \begin{Bmatrix} (\text{Had on } A)^{-1} \\ (\text{NOT on } A)^{-1} \end{Bmatrix} = \begin{Bmatrix} \text{Had on } A \\ \text{NOT on } A \end{Bmatrix}$$

Check:

$$|0\rangle \xmapsto{\text{Had}} |+\rangle = \sqrt{\tfrac{1}{2}}\,|0\rangle + \sqrt{\tfrac{1}{2}}\,|1\rangle \xmapsto{\text{NOT}} \sqrt{\tfrac{1}{2}}\,|0\rangle + \sqrt{\tfrac{1}{2}}\,|1\rangle = |+\rangle$$

$$|1\rangle \xmapsto{\text{Had}} |-\rangle = \sqrt{\tfrac{1}{2}}\,|0\rangle - \sqrt{\tfrac{1}{2}}\,|1\rangle \longmapsto -\sqrt{\tfrac{1}{2}}\,|0\rangle + \sqrt{\tfrac{1}{2}}\,|1\rangle = -|-\rangle$$

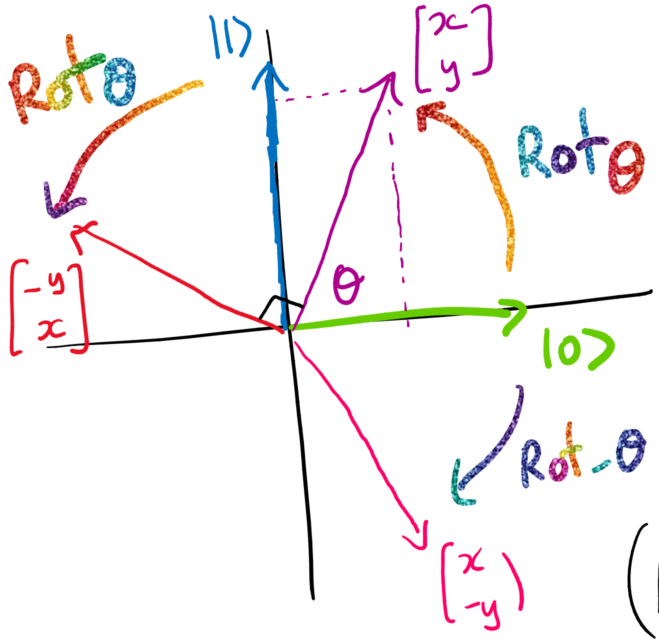
[I know, this notation seems crazy!]
Negative of the vector $|-\rangle$!



$Rot_{45°}$ ✓

$$Rot^{\dagger}_{-45°} = \begin{array}{cc} \begin{bmatrix} \sqrt{\tfrac{1}{2}} & -\sqrt{\tfrac{1}{2}} \\ \sqrt{\tfrac{1}{2}} & \sqrt{\tfrac{1}{2}} \end{bmatrix} \\ \begin{array}{cc} \quad |+\rangle & \qquad -|-\rangle \end{array} \end{array}$$

[So, with the instructions we "have" so far, NOT & Had, we can build $Rot_{45°}$, hence "rotation by any multiple of 45°".
What about rotations by other angles?
Can't build from NOT & Had, but....]



$$Rot_\theta = \begin{array}{cc} & \begin{array}{cc} |0\rangle & \text{in} \quad |1\rangle \end{array} \\ \text{out} & \begin{bmatrix} x & -y \\ y & x \end{bmatrix} \end{array}$$

$$(Rot_\theta)^{-1} = Rot_{-\theta} = \begin{array}{c} \begin{array}{cc} |0\rangle & |1\rangle \end{array} \\ \begin{bmatrix} x & y \\ -y & x \end{bmatrix} \end{array} = (Rot_\theta)^\dagger$$

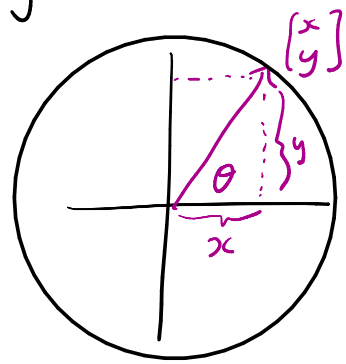
[So Rot_θ is unitary, reverse = inverse.
∴ Laws of Physics say it is an allowable transformation on qubits.
So, okay...] Rot_θ instruction now allowed! (for any θ)

[Different physical implem. depending on how qubit implem'd.
E.g. photon polarization: pass thru a sugar-water solution of certain concentration, e.g.!]

$$\text{Rot}_\theta = \begin{bmatrix} x & -y \\ y & x \end{bmatrix}, \text{ what are } x \ \& \ y?$$

$x = \cos\theta$

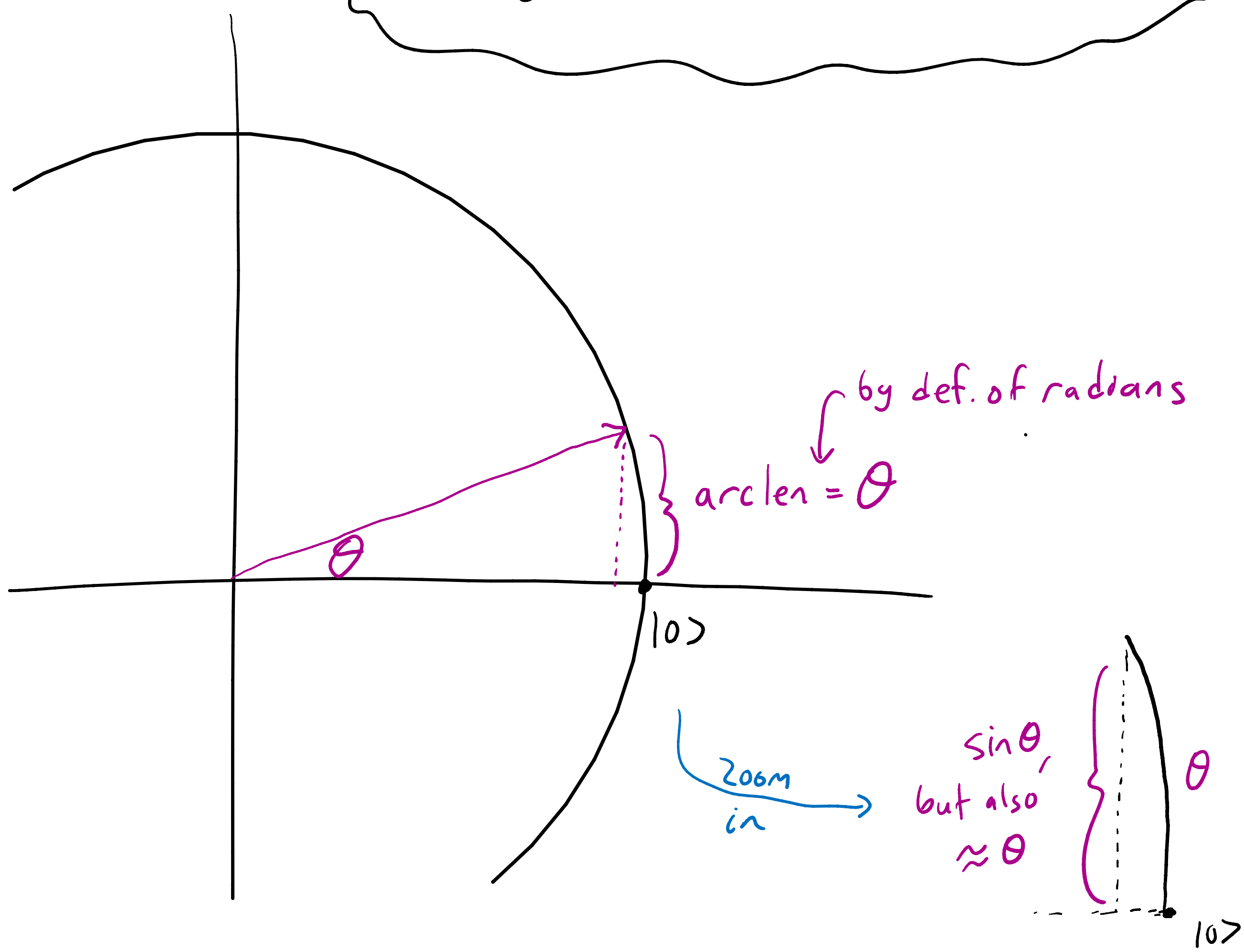
$y = \sin\theta$



[This is really just the definition of cos, sin.]

[Most important thing to know about cos & sin ...]

$$\sin\theta \approx \theta \text{ if } \theta \text{ is small}$$

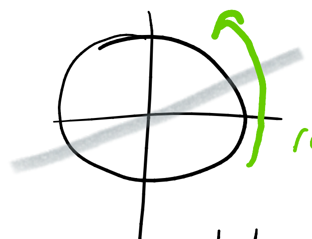


Facts about geometry of $\mathbb{R}^2$:

- Every unitary matrix is a rotation or reflection

  ⟦Also true in higher dims! All quantum operations are rotations / reflections.⟧

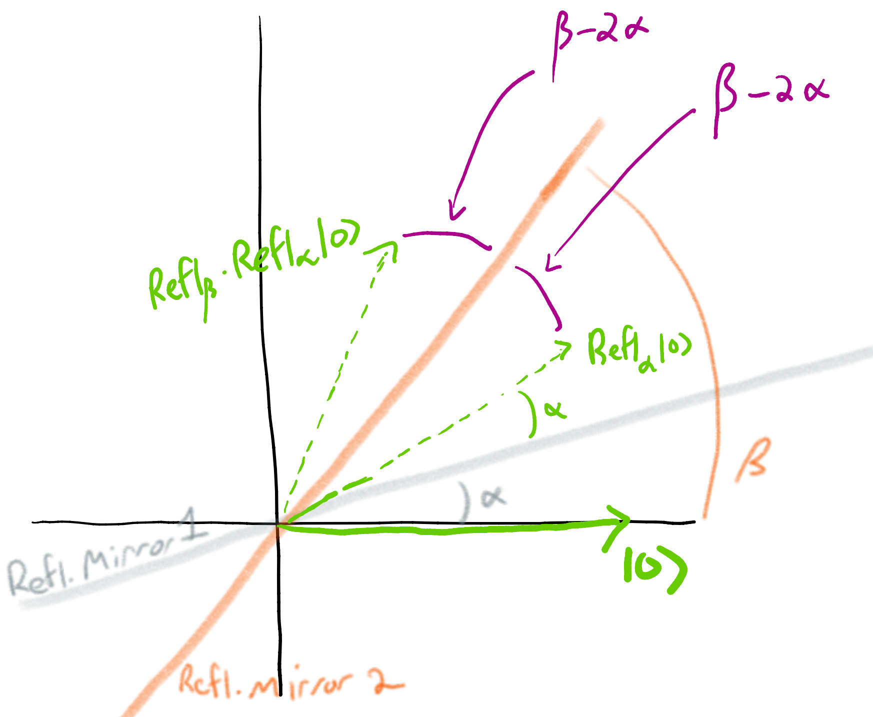
- A reflection is a rotation if you allow ≥ 1 more dimension. ⟦1 more qubit = twice as many dims⟧



reflect by rotating thru 3rd dim.!

- Conversely, every rotation is combo of 2 reflections.

  ⟦We saw this with $Had \cdot NOT = Rot_{45°}$.⟧



Final rotation angle of $Refl_\beta \cdot Refl_\alpha |0\rangle$ is

$$\beta + (\beta - 2\alpha)$$
$$= 2\beta - 2\alpha$$
$$= 2(\beta - \alpha).$$

[More on angles...]

[Say you have vector...] $|v\rangle = \begin{bmatrix} x \\ y \end{bmatrix} = x|0\rangle + y|1\rangle$

Q: What is its "0th coord"?

A: x, obviously.

Also "dot product (inner product)" of

$\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} x \\ y \end{bmatrix}$, $\quad 1 \cdot x + 0 \cdot y = x.$

Equivalently, as a $(1\times 2)(2\times 1)$ matrix mult.:

$\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 1 \cdot x + 0 \cdot y.$

$|0\rangle$

$|0\rangle^\dagger$, the reverse of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$
(transpose)

[[Subtlety: mathematicians freely typecast between #'s & 1x1 matrices.]]

Dirac Notation feature:

"$\langle blah|$" means $|blah\rangle^\dagger$

"$\langle \cdot |$", pronounced "bra", signifies row vector

$\therefore$ 0^{th} coord of $|v\rangle$ is $\langle 0| \cdot |v\rangle$,

[[$\cdot$ for multiplication often dropped.
Dirac: don't write both bars in $\langle 0||v\rangle$]]

denoted just $\langle 0|v\rangle$.

[[Final glorious feature of notation. $\langle u|v\rangle$ is inner product between $|u\rangle, |v\rangle$. "bra-ket"]]

$\therefore$ 1^{th} coord of $|v\rangle$ is $\langle 1|v\rangle$.

Printing qubit in state $|v\rangle$:

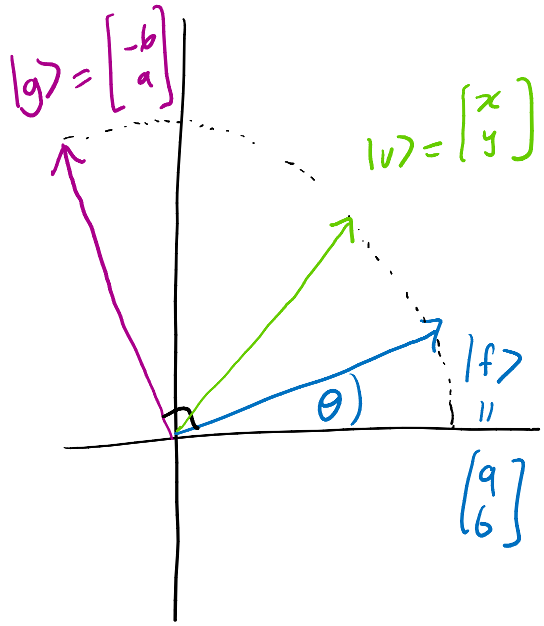
Output "0" w. prob. $|\langle 0|v\rangle|^2$
"1" w. prob. $|\langle 1|v\rangle|^2$.

Say $|f\rangle$ is another state vector, $|f\rangle = \begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2$.

What is $\langle f|v\rangle$? $[a \ b]\begin{bmatrix} x \\ y \end{bmatrix} = ax + by$, yes.

But what <u>is</u> this #? [[I.e., why compute it?]]

[[Since we know a good meaning for $\langle 0|vec\rangle$, namely $|vec\rangle$'s 0^{th} coord, let's try to get $|0\rangle$ in picture.]]



Say $|f\rangle = Rot_\theta\,|0\rangle$.

$$\therefore\ Rot_\theta = \begin{array}{cc} & \begin{array}{cc}|0\rangle \text{ in} & |1\rangle\end{array} \\ \text{out} & \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \\ & \begin{array}{cc}\uparrow |f\rangle & \uparrow |g\rangle\end{array}\end{array}$$

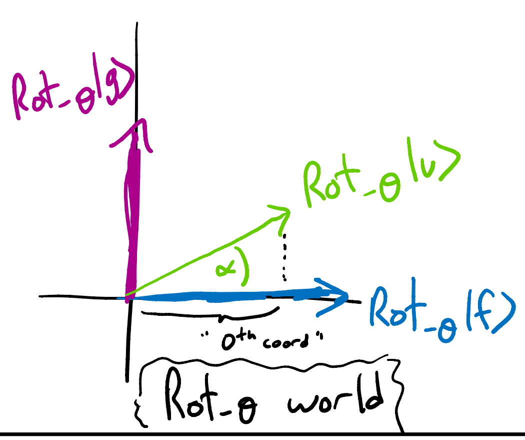
$$\langle f| = |f\rangle^\dagger = [a \;\; b]$$

$$= (Rot_\theta \cdot |0\rangle)^\dagger$$

[used $(P.Q)^\dagger = Q^\dagger . P^\dagger$] $\longrightarrow$ $= \langle 0| \cdot Rot_\theta^\dagger$

$$= \langle 0| \cdot Rot_{-\theta}$$

$$\therefore\ \langle f|v\rangle = \langle 0|\, Rot_{-\theta}\,|v\rangle = 0^{th} \text{ coord of } Rot_{-\theta}|v\rangle!$$



$\therefore\ \langle f|v\rangle =$ "$|v\rangle$'s coord on $|f\rangle$ in $(|f\rangle, |g\rangle)$ basis"

$=$ len. of $|v\rangle$'s projection to $|f\rangle$

$= \cos\alpha$, $\alpha =$ angle of $|v\rangle$ from $|f\rangle$

Similarly, $\langle g|v\rangle = \langle 1|\,Rot_{-\theta}\,|v\rangle$

$= 1^{th}$ coord of $Rot_{-\theta}|v\rangle$

$= |v\rangle$'s coord on $|g\rangle$ in $(|f\rangle, |g\rangle)$ basis

$= \sin\alpha$

$= \cos(90° - \alpha)$

$= \cos($angle of $|v\rangle$ to $|g\rangle)$

(These facts all for $|v\rangle$, $|f\rangle$, $|g\rangle$ unit-length.)

Generally: $\langle p|q\rangle = \| \, |p\rangle \| \cdot \| \, |q\rangle \| \cdot \cos(p,q \text{ angle})$

"Cosine Law"

[[Finally, let's talk more about...]]

Measuring one qubit A

[[assume $x, y \in \mathbb{R}$]]

"Print All" on qubit in state $|v\rangle = x|0\rangle + y|1\rangle$

↳ output "0" w.prob $x^2 = \langle 0|v\rangle^2$

output "1" w.prob $y^2 = \langle 1|v\rangle^2$,

and qubit "destroyed".

<u>Rem:</u> could always "remake" A:

- Make A

if Output = 1, • Add 1 to A

In reality:

- Called "Measure", not "Print"
- Qubit always "remade"

[[Or rather — just not destroyed. Physical qubits can't just disappear.]]

"if readout "0", state of A "collapses" to $|0\rangle$;

" " "1", " " " " to $|1\rangle$ "

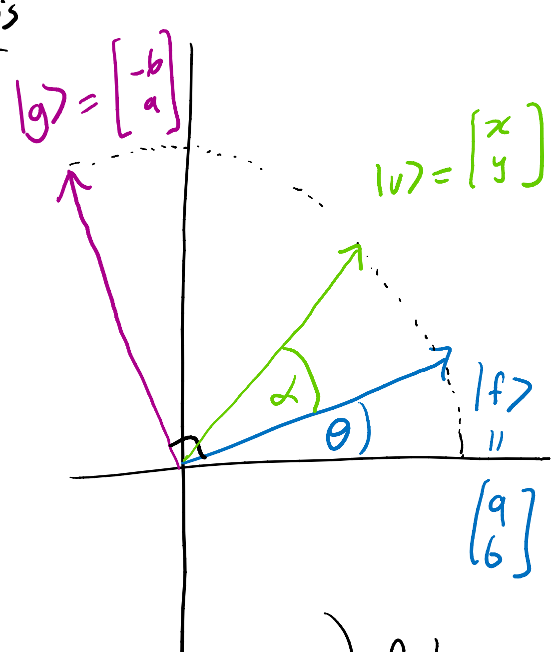
[[Anyway, we mainly only measure at end of algs, so needn't worry so much about post-measurement state.]]

[Finally, a very important new concept...]

Measuring In A Different Basis

def Algorithm (A):

1. $Rot_{-\theta}$ on A
2. Measure A



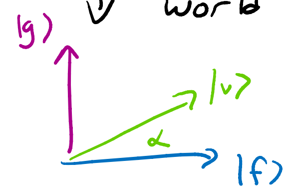
Q: If A's state is $|v\rangle$, what does this do?

Prob[output "0" in line 2] =

$\left(|v\rangle\text{'s coord on } |f\rangle \text{ in } (|f\rangle, |g\rangle) \text{ basis}\right)^2$

$= (\cos\alpha)^2$. → and A "collapses" to $|0\rangle$

$Rot_{-\theta}$ world



Pr[output "1" in line 2] =

$\left(|v\rangle\text{'s coord on } |g\rangle \text{ in } (|f\rangle, |g\rangle) \text{ basis}\right)^2$

$= (\sin\alpha)^2 = \cos(90° - \alpha)^2$. → and A "collapses" to $|1\rangle$

[Now it's logical to add 2 more lines...]

3. If output "0", Print("f")
   Else if output "1", Print("g")
4. Rot_{θ} on A // moves $|0\rangle$ to $|f\rangle$, $|1\rangle$ to $|g\rangle$.

Lines 1...4 called "Measuring in $(|f\rangle, |g\rangle)$ basis"

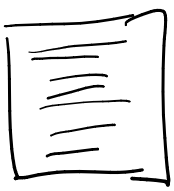
"Measuring in $(|f\rangle, |g\rangle)$ basis" summary:

When applied to qubit in state $|v\rangle$:

$\Pr[\text{printing "f"}] = \langle f|v\rangle^2 \rightarrow$ state collapses to $|f\rangle$

$\Pr[\text{printing "g"}] = \langle g|v\rangle^2 \rightarrow$ state collapses to $|g\rangle$

[["usual" measurement is measuring in "standard" basis $(|0\rangle, |1\rangle)$]]



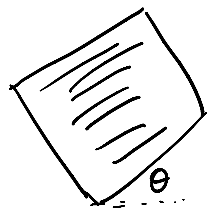
H.P.F

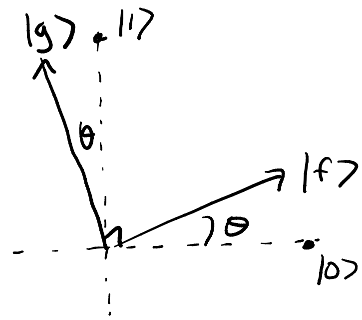
Horizontally polarizing filter

$|0\rangle$ - passes

$|1\rangle$ - blocked/heat

physically rotate by θ

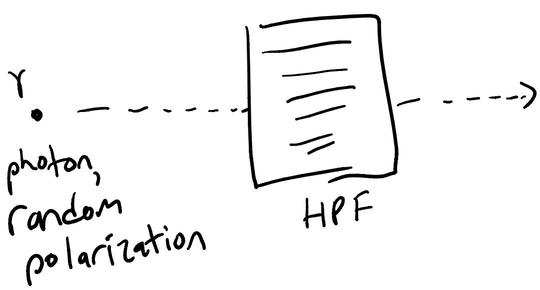




Measures in $(|f\rangle, |g\rangle)$ basis:

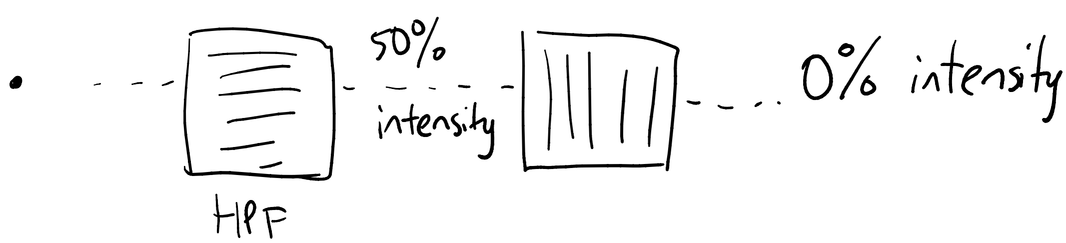
"readout" "f" $\rightarrow$ photon passes thru, new state $|f\rangle$

"readout" "g" $\rightarrow$ blocked, heat

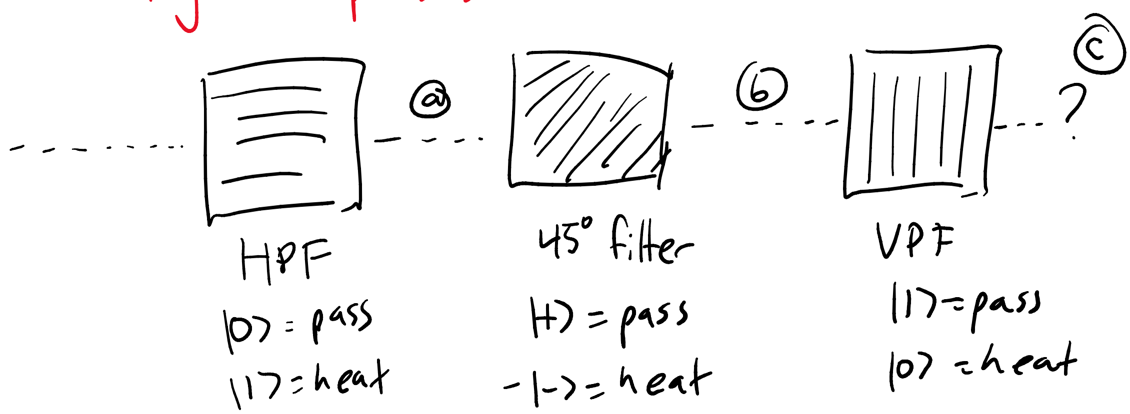


$\Pr[\text{passes thru}] = 50\%$

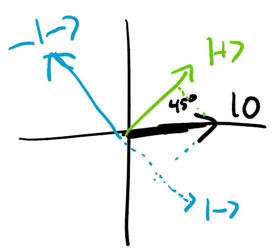
If so, new state is $|0\rangle$.



[[Now suppose you interpose a 45°-rotated filter in middle. Childlike intuition: previously all light was blocked, then you only further added a barrier. Still no light passes thru??]]



At (a): 50% intensity, photons deterministically $|0\rangle$.



Measure in $(|+\rangle, -|-\rangle)$ basis.

$$\text{Prob}[\text{output "+"}] = (\cos 45^\circ)^2 = \left(\sqrt{\tfrac{1}{2}}\right)^2 = \frac{1}{2}$$

↳ passes thru, state collapses to $|+\rangle$

At (b): 25% intensity, states all $|+\rangle = \sqrt{\tfrac{1}{2}}\,|0\rangle + \sqrt{\tfrac{1}{2}}\,|1\rangle$

$$\text{Prob}[\text{measured at VPF to "1"}] = \left(\sqrt{\tfrac{1}{2}}\right)^2 = \tfrac{1}{2}.$$

↳ passes thru

$\therefore$ At (c): 12.5% intensity!! [[Light now passes thru!! All photons here in state $|1\rangle$.]]