Lecture 13 — Adding & Deleting Qubits

[Want to start by telling you a dirty secret about...] Probability Theory:

Week 1: Computing w/ probability trees etc.

Week 2: "New definition: two random vars A & B are <u>independent</u> if $\forall$ vals a, b,

$$\Pr[A=a \ \& \ B=b] = \Pr[A=a] \cdot \Pr[B=b]."$$

E.g.: say we [do some experiment/probabilistic code] involving bits A, B manage to compute:

a	b	$\Pr[AB=ab]$
0	0	.63
0	1	.07
1	0	.27
1	1	.03

"Hey look!

$= .7 \times .9$

$= .7 \times .1$

$= .3 \times .9$

$= .3 \times .1$

So A, B are independent!"

[said no one ever!!]

[This is not how the life of a probabilist works!]

Week 3: ⟦You're solving some problem ...⟧

analyzing

```
Code1:
    Make A
    Noise A
    Make B
    Fade A
    Fade B
    Noise A
```

// some old prob. computing
// instructions; exact
// defs not important

Q: "What's joint prob. of A, B?"

A: First, ⟦Code1 written weirdly. Note that no instructions involve A & B together. It's like Alice is operating on A in one room, Bob on B in another room, and really the exact timing of who does what when doesn't matter. So...⟧ equiv. to:

```
Code 2:
    Make A
    Noise A
    Fade A
    Noise A

    Make B
    Fade B
```

⟦That <u>is</u> true,
Code1 & Code2
<u>are</u> equivalent...⟧

Then... [[you'd say "those two blocks creating A & creating B don't affect each other. So...]] clearly A & B are independent,

so can
- compute probs of $A=0,1$
- compute probs of $B=0,1$,
- multiply to get all 4 $\Pr[AB=ab]$.

[[This is all true. Except...]]

AHA! [[J'accuse! You cheated! According to the definition, you first have to <u>know</u> all the $\Pr[AB=ab]$ vals, check if they're each $= \Pr[A=a] \cdot \Pr[B=b]$, and only <u>then</u> can you say they're independent. Not the other way around!

Yet probabilists do the above all the time. What's going on? Well, they never tell you the dirty little secret of independence...]]

Dirty Secret Theorem:

Say 'Alice' forms a probabilistic bit A, with probabilities $\begin{bmatrix} r \\ s \end{bmatrix} \begin{matrix} (A=0) \\ (A=1) \end{matrix}$

Say 'Bob', with completely separate code, forms a probabilistic bit B, with probabilities $\begin{bmatrix} x \\ y \end{bmatrix} \begin{matrix} (B=0) \\ (B=1) \end{matrix}$

— no interactions/code involving both A & B.

[even thru other variables]

Then the joint distribution of A, B will be $\begin{bmatrix} rx \\ ry \\ sx \\ sy \end{bmatrix} \begin{matrix} (AB=00 \\ 01 \\ 10 \\ 11) \end{matrix}$, and you may say the phrase, "A & B are independent".

[[Note: this is a (simple to prove) theorem about how you'll always get these same 4 numbers rx, ry, sx, sy if you rearrange certain prob. code & hence probability trees. Because of this, it's exactly equally true for quantum instructions & amplitude trees!]]

SAME THEOREM IF "PROBS" CHANGED TO "AMPLITUDES"

And we say "unentangled", not "independent".

Note: we just define "unentangled," not "entangled"
"Entangled" is just "not unentangled."

More general fact:

If A_1, A_2 are prob. or qu-bits with probs/ampls

$$|u\rangle := \begin{bmatrix} p_{00} \\ p_{01} \\ p_{10} \\ p_{11} \end{bmatrix} \begin{array}{c} A_1 A_2 \\ 00 \\ 01 \\ 10 \\ 11 \end{array}$$

& $B_1, B_2 B_3$ are prob/qu-bits with probs/ampls

$$|v\rangle := \begin{bmatrix} q_{000} \\ q_{001} \\ \vdots \\ q_{111} \end{bmatrix} \begin{array}{c} B_1 B_2 B_3 \\ 000 \\ 001 \\ \vdots \\ 111 \end{array}$$

& they're prepared separately, no interaction, then they are independent/unentangled with joint state

[vector of height $4 \times 8 = 32$] →

$$\begin{bmatrix} p_{00} \cdot q_{000} \\ p_{00} \cdot q_{001} \\ \vdots \\ p_{10} \cdot q_{011} \\ \vdots \\ p_{11} \cdot q_{111} \end{bmatrix} \begin{array}{c} A_1 A_2\, B_1 B_2 B_3 \\ 0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,1 \\ \vdots \\ 1\,0\,0\,1\,1 \\ \vdots \\ 1\,1\,1\,1\,1 \end{array}$$

← notation for this vector is $|u\rangle \otimes |v\rangle$

[Pronounced "tensor". \otimes in LaTeX]

[Also OK for unnormalized states]

Q: What is $|10\rangle \otimes |011\rangle$? [[It answers the question, if Alice prepares 10 for sure, Bob prepares 011 for sure, what state do they have jointly?]]

A: $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ ← 10011 row

$\therefore |10\rangle \otimes |011\rangle = |10011\rangle$.

Nice! [[notational felicity]]

Say A has state $\begin{bmatrix} r \\ s \end{bmatrix} = r|0\rangle + s|1\rangle$,

B " " $\begin{bmatrix} x \\ y \end{bmatrix} = x|0\rangle + y|1\rangle$.

Then joint state is $(r|0\rangle + s|1\rangle) \otimes (x|0\rangle + y|1\rangle)$

$$= rx|00\rangle + ry|01\rangle + sx|10\rangle + sy|11\rangle.$$

Like we applied "FOIL" (law for distributing) and used:

- $\otimes$ acts like <u>non</u>com<u>mutative</u> mult.
- $|a\rangle \otimes |b\rangle = |ab\rangle$
- scalars commute with everything.

[[Remark: $\otimes$ is also associative.]]

eg.: Alice makes A in state $|+\rangle$,
Bob makes B " " $|0\rangle$.
They bring them together. Joint state is unentangled: $|+\rangle \otimes |0\rangle$

$$= \left(\sqrt{\tfrac{1}{2}}\,|0\rangle + \sqrt{\tfrac{1}{2}}\,|1\rangle\right) \otimes |0\rangle$$

$$= \sqrt{\tfrac{1}{2}}\,|00\rangle + \sqrt{\tfrac{1}{2}}\,|10\rangle.$$

Now say they perform "Add A to B".
[[Note: they must be physically collocated to do this!]]

New state: $\sqrt{\tfrac{1}{2}}\,|00\rangle + \sqrt{\tfrac{1}{2}}\,|11\rangle = \begin{bmatrix} \sqrt{1/2} \\ 0 \\ 0 \\ \sqrt{1/2} \end{bmatrix}$

[This is a famous state, we'll study it a lot next lecture!]

Q: Is this state unentangled?
[Uh... probably not? Certainly the Dirty Secret Theorem doesn't let you say it is. But could it be, by luck?]

A: No. AFSOC it's $\begin{bmatrix} r \\ s \end{bmatrix} \otimes \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} rx \\ ry \\ sx \\ sy \end{bmatrix}$. Then

① $rx = \sqrt{1/2}$, ② $ry = 0$, ③ $sx = 0$, ④ $sy = \sqrt{1/2}$.

But ①·④ $\Rightarrow rsxy = 1/2$, ②③ $\Rightarrow rsxy = 0$, $\Rightarrow\Leftarrow$.

Interlude: Unnormalized States

Say A, B are qubits with joint state

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = w|00\rangle + x|01\rangle + y|10\rangle + z|11\rangle$$

~~Print All~~ Measure All: what happens?

[Well, first ask if this state is normalized...]

If normalized: $\Pr[\text{readout "00"}] = w^2$, collapses to $|00\rangle$

$\sim\sim$ "01" x^2, $\sim\sim$ $|01\rangle$

etc.

If unnormalized: $\Pr[\text{readout "00"}] = \dfrac{w^2}{w^2+x^2+y^2+z^2}$, collapses to $|00\rangle$

$\sim\sim$ "01" $= \dfrac{x^2}{w^2+x^2+y^2+z^2}$, etc.

Why? Well can first normalize:

mult by scalar c so that $c\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}$ has

sum of |squares| $= 1$.

$$c = \frac{1}{\sqrt{w^2+x^2+y^2+z^2}}$$

E.g. famous entangled state : $|00\rangle + |11\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ ⊛
(unnormalized)

Could also express as $\begin{bmatrix} 1/2 \\ 0 \\ 0 \\ 1/2 \end{bmatrix}$ (unnormalized).

Or as $|v\rangle = \begin{bmatrix} \sqrt{1/2} \\ 0 \\ 0 \\ \sqrt{1/2} \end{bmatrix}$ (normalized).

Q: What about as $-|v\rangle = \begin{bmatrix} -\sqrt{1/2} \\ 0 \\ 0 \\ -\sqrt{1/2} \end{bmatrix}$? [[Hey, I just multiplied ⊛ by $c = -\sqrt{1/2}$ and got a vec. with $\Sigma\, \text{ampl}^2 = 1$.]]

A: YES! It's ok!

<u>Fact</u>: $|v\rangle$ and $-|v\rangle$ are equally valid math representations of the same physical state.

If I promise you $|\text{test}\rangle = |v\rangle$ or $-|v\rangle$, no physical experiment you ever do can tell difference.

[[Apply a unitary?]] $|v\rangle \overset{U}{\longmapsto} U|v\rangle \qquad -|v\rangle \overset{U}{\longmapsto} U(-|v\rangle) = -U|v\rangle$

[[still negatives of each other!]]

[[Measure all?]] Measure: $-$ sign gets squared away.

[[You learned in 3rd grade...]]

as fractions, $\frac{2}{3}$, $\frac{20}{30}$, $\frac{8}{12}$, $\frac{-2}{-3}$, $\frac{2i}{3i}$

are all the same #. OK to multiply top/bottom by same nonzero scalar.

So too with quantum states!

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} \ \& \ \begin{bmatrix} cw \\ cx \\ cy \\ cz \end{bmatrix}$$ are equiv. (unnormalized) representations of same quantum state

for any $c \neq 0$.

Measuring one out of several qubits

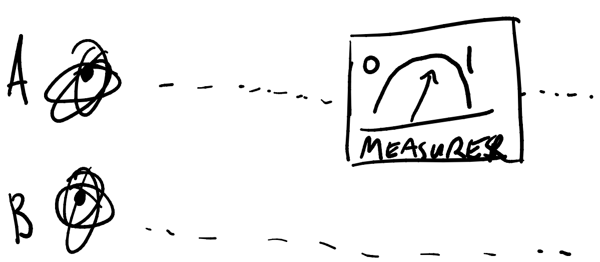
Say A,B are qubits with joint normalized

$$\text{state} \quad \begin{array}{c c} & AB \\ \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} & \begin{array}{c} 00 \\ 01 \\ 10 \\ 11 \end{array} \end{array} = w|00\rangle + x|01\rangle + y|10\rangle + z|11\rangle,$$

$$w^2 + x^2 + y^2 + z^2 = 1.$$

[actually $|w|^2 + |x|^2 + |y|^2 + |z|^2 = 1$]

[We know what happens if you "Measure All". But these are, physically, like 2 photons or 2 electrons or something. Nothing stops you from putting just 1 of them into a measuring device...]



Now what?

- Readout can only show "0" or "1"
- Repeatedly measuring same qubit always gives same readout.

[Well, you can't deduce what happens based on rules I've told you so far. I actually have to tell you a new rule of Quantum Mechanics...]

[[Luckily, the only possible plausible rule is indeed the rule.]]

$$AB: \begin{array}{cc} & AB \\ \left[\begin{array}{c} w \\ x \\ y \\ z \end{array}\right] & \begin{array}{c} 00 \\ 01 \\ 10 \\ 11 \end{array} \end{array}$$

Measuring A:

$\text{Prob}[\text{readout} = \text{"0"}] = w^2 + x^2$ [[sum of $|\text{amp}|^2$ on "pieces" with $A=0$]]

$\text{Prob}[\text{readout} = \text{"1"}] = y^2 + z^2.$ [[✓ these two #'s add to 1]]

Joint state "collapses" to

$w|00\rangle + x|01\rangle$

[[the "pieces" with $A=0$]]

Joint state "collapses" to

$y|10\rangle + z|11\rangle$

[["pieces" with $A=1$]]

These new states are unnormalized.
Can normalize if you want.

Say you got readout "$A = 0$". (unnormalized) state now $w|00\rangle + x|01\rangle$

Say you now Measure B. What happens?

[[Follow same rule!]] $\text{Prob}[\text{readout "0"}] = \left(\frac{w^2}{w^2+x^2}\right),$
joint state collapses to $w|00\rangle$. (unnormalized)

Note: Overall prob. to see $A=0$ then $B=0$

is $(w^2+x^2)\cdot\dfrac{w^2}{w^2+x^2} = w^2$, and state

collapses to $\underset{\text{(unnorm'd)}}{w|00\rangle} \equiv \underset{\text{(norm'd)}}{|00\rangle}$.

Same as if we did Measure All!

Easy to check:

$$\boxed{\begin{array}{l}\text{Measure } A\\ \text{Measure } B\end{array}} \equiv \boxed{\begin{array}{l}\text{Measure } B\\ \text{Measure } A\end{array}} \equiv \boxed{\text{Measure All}}$$

e.g.: Say we Measure B, then A.

$\text{Prob}[\text{see } B=1, A=0]$?

$$\begin{array}{cc} & AB \\ \left[\begin{array}{c} w \\ x \\ y \\ z \end{array}\right] & \begin{array}{c} 00 \\ 01 \\ 10 \\ 11 \end{array}\end{array}$$

Measure B: $\text{Prob}[\text{readout } 1] = x^2+z^2$

& if so, state collapses to $x|01\rangle + z|11\rangle$

Measure A: $\text{Prob}(\text{readout } 0) = \dfrac{x^2}{x^2+z^2}$, & if so collapse to $x|01\rangle$.

Overall: $\text{Prob}[\text{see } A=0, B=1] = (x^2+z^2)\left(\dfrac{x^2}{x^2+z^2}\right) = x^2$

& if so, collapse to $x|01\rangle \equiv |01\rangle$.

[[Same as if Measure All!]]

FINAL KEY POINT: Measuring a qubit <u>disentangles</u> it.

$$\begin{array}{cc} & AB \\ \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} & \begin{array}{c} 00 \\ 01 \\ 10 \\ 11 \end{array} \end{array}$$

Say we measure A, see readout "1".

New state: $y|10\rangle + z|11\rangle$

$$\ldots = \underset{A}{|1\rangle} \otimes \underset{B}{(y|0\rangle + z|1\rangle)}$$

A is <u>unentangled</u> with B.

↳ can always "factor" out A's $|1\rangle$ because it's 1 in every "piece".

Or had we measured B, seen "0":

new state: $w|00\rangle + y|10\rangle = \underset{A}{(w|0\rangle + y|1\rangle)} \otimes \underset{B}{|0\rangle}$

Equivalent as if B "deleted", then separately "remade" as $|0\rangle$.

This is the <u>one</u> circumstance (an unentangled qubit) where it's okay to drop/delete/forget a qubit from a joint state.

[[Can pretend like it was never "remade".]]