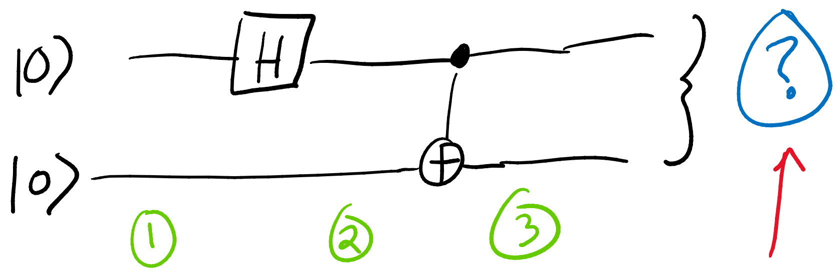
Lecture 14 — A Maximally Entangled Pair of Qubits

[We've discussed 1 qubit, & 2 unentangled qubits....
Finally, we'll get to 2 entangled qubits....]

- Make A, B
- H on A
- CNOT A, B
  // add A to B

$\equiv$



[The most famous entangled state]

State at (1) $|00\rangle = |0\rangle \otimes |0\rangle$

(2) $|+\rangle \otimes |0\rangle = \left(\sqrt{\tfrac{1}{2}}|0\rangle + \sqrt{\tfrac{1}{2}}|1\rangle\right) \otimes |0\rangle$

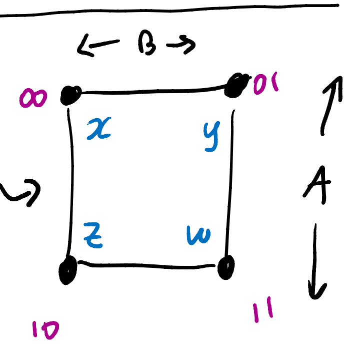
(still unentangled) $= \sqrt{\tfrac{1}{2}}|00\rangle + \sqrt{\tfrac{1}{2}}|10\rangle$

(3) $\sqrt{\tfrac{1}{2}}|00\rangle + \sqrt{\tfrac{1}{2}}|11\rangle = \sqrt{\tfrac{1}{2}} \cdot \begin{array}{cl} & AB \\ \left[\begin{array}{c} 1 \\ 0 \\ 0 \\ 1 \end{array}\right] & \begin{array}{l} 00 \\ 01 \\ 10 \\ 11 \end{array} \end{array}$

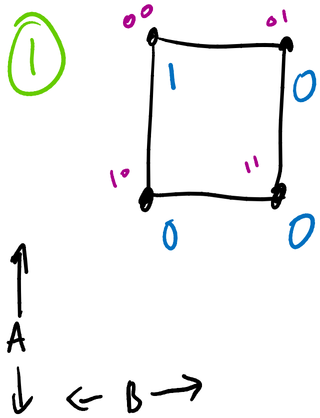
[note: could drop the $\sqrt{\tfrac{1}{2}}$ everywhere in this lecture, for unnormalized convenience]

[Now that we have 2 qubits = 4 dims, can't picture vectors geometrically so easily. Best bet is to "plot" 4 or 8 amplitudes on square/cube: ◻ or ⧈]

$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \rightsquigarrow$

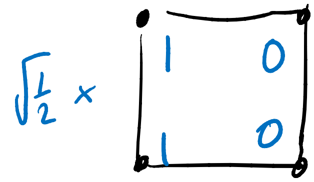


1



H on A $\longrightarrow$

[Pair up so only differing coord is "A" = vert. Then do "Add & Diff" $\times \sqrt{\tfrac{1}{2}}$]



2

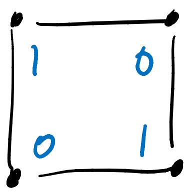
CNOT

// Add A to B

(swaps $|10\rangle$, $|11\rangle$)

3

$$\sqrt{\tfrac{1}{2}}\,|00\rangle + \sqrt{\tfrac{1}{2}}\,|11\rangle \; = \; \sqrt{\tfrac{1}{2}} \times$$



"Maximally Entangled Pair" ≡ "Bell pair"

≡ "EPR pair"

Einstein-Podolsky-Rosen

$|MEP\rangle$

[not standard notation]

[Sorta standard name]

[These 4 people had some famous thought experiments about this state.]

[One can try to quantify, for an entangled pair of qubits, just "how" entangled it is. By any measure, $|MEP\rangle$ is the most entangled.
But, we won't try to quantify entanglement.
BTW: there are some other states which are equally maximally entangled, so this isn't THE max entangled pair. But we'll still call it M.E.P.]

[$|MEP\rangle$ has some famously interesting properties. Let's explore....]

Say Alice & Bob make $|MEP\rangle$.
Then....

[Convention: Alice "owns" first qubit A, Bob "owns" B.]

Scenario 1

[Add 1 to A]

Alice does NOT on A:

$$\sqrt{\tfrac{1}{2}}\,|00\rangle + \sqrt{\tfrac{1}{2}}\,|11\rangle$$

$$\mapsto \sqrt{\tfrac{1}{2}}\,|10\rangle + \sqrt{\tfrac{1}{2}}\,|01\rangle$$

$$= \sqrt{\tfrac{1}{2}}\,|01\rangle + \sqrt{\tfrac{1}{2}}\,|10\rangle$$

Scenario 2

Bob does NOT on B:

$$\sqrt{\tfrac{1}{2}}\,|00\rangle + \sqrt{\tfrac{1}{2}}\,|11\rangle$$

$$\mapsto \sqrt{\tfrac{1}{2}}\,|01\rangle + \sqrt{\tfrac{1}{2}}\,|10\rangle$$

Equal! Hmm.

[Oddly, Alice doing NOT in her room is equivalent to Bob doing NOT in his room.
Is this weird? I dunno, not that weird I guess.]

$\Rightarrow$
- Start at $|MEP\rangle$
- Alice does NOT $\longrightarrow$ equiv, Bob does NOT
- Bob does NOT

these cancel

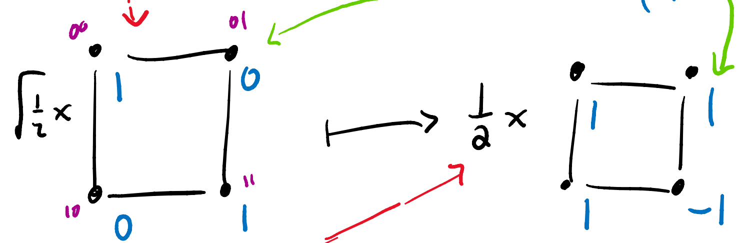
Leaves state as $|MEP\rangle$

[Let's try this game with HAD in place of NOT....]

Start with $|MEP\rangle$.

Scenario 1:

Alice does H on A:



Add&Diff $\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\-1\end{pmatrix}$

Add&Diff $\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}$

[This $\frac{1}{2}$ is $\sqrt{\frac{1}{2}} \times \sqrt{\frac{1}{2}}$

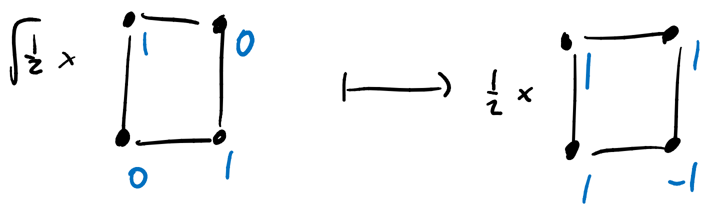
from $|MEP\rangle$

$H = \sqrt{\frac{1}{2}} \times$ Add&Diff]

Scenario 2:

Bob does H on B:

[Similar to above, but since dims. are $\updownarrow A$, $\leftarrow B \rightarrow$, we do Add&Diff "horizontally", not "vertically"]



Same again! From $|MEP\rangle$,

Alice doing HAD on A $\equiv$ Bob doing HAD on B.

[again] $\Rightarrow$

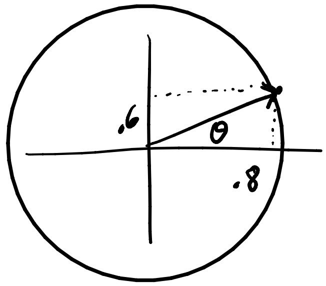
- $|MEP\rangle$
- H on A $\longrightarrow$ equiv to H on B
- H on B

these 2 cancel

Produces $|MEP\rangle$ again.

[What else can we do on 1 qubit? Rotate....]

Fix this θ:



$\theta \approx 37°$.

$Rot_\theta \begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}.8\\.6\end{bmatrix}$, $Rot_\theta \begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}-.6\\.8\end{bmatrix}$

Say Alice does Rot_θ to A.

$$\sqrt{\tfrac{1}{2}} \times \begin{array}{c} \begin{array}{cc} 00 & 01 \end{array} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ \begin{array}{cc} 10 & 11 \end{array} \end{array} \xmapsto{Rot_\theta \text{ on } A} \sqrt{\tfrac{1}{2}} \times \begin{bmatrix} .8 & -.6 \\ .6 & .8 \end{bmatrix}$$

$|MEP\rangle$

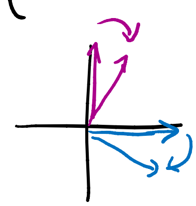
[Do Rot_θ on each column]

[If instead we did Rot_θ on B, it'd go horizontally again.

Would get $\sqrt{\tfrac{1}{2}} \times \begin{bmatrix} .8 & .6 \\ -.6 & .8 \end{bmatrix}$. Not quite the same. To match, we need the operation that does...]

$$\begin{bmatrix}1\\0\end{bmatrix} \mapsto \begin{bmatrix}.8\\-.6\end{bmatrix}, \quad \begin{bmatrix}0\\1\end{bmatrix} \mapsto \begin{bmatrix}.6\\.8\end{bmatrix}$$

$$Rot_\theta^\dagger = Rot_{-\theta}$$



So from $|MEP\rangle$, Alice doing Rot_θ on A is equiv. to Bob doing $Rot_{-\theta}$ on B.

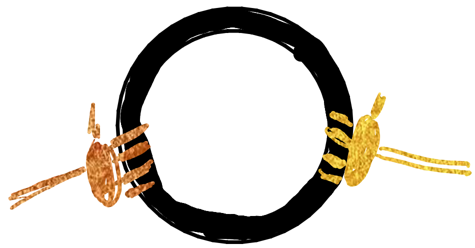
[We've now pretty much proved (by example):]

Theorem: Starting from $|MEP\rangle$ on A, B,
Alice doing U on A
$\equiv$ Bob doing $U^\dagger$ on B

[We saw $U = NOT = U^\dagger$,
$U = H = U^\dagger$,
$U = Rot_\theta$, $U^\dagger = Rot_{-\theta}$]

<u>Mnemonic:</u> $|MEP\rangle$ is like a shared steering wheel.

Alice's right hand U



Bob's left hand $U^\dagger$

These do the same thing

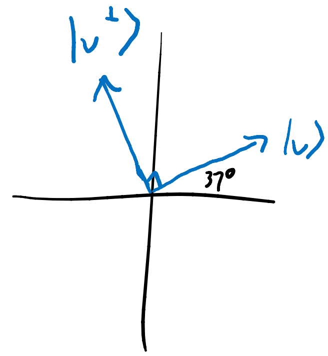
<u>Observe:</u> From $|MEP\rangle$: • Alice does U ⟶ equiv Bob does $U^\dagger$

(then) • Bob does U — these cancel!

Gives $|MEP\rangle$ back!

E.g. let $U = Rot_{37°}$, let $|v\rangle = U|0\rangle$

$|v^\perp\rangle = U|1\rangle$.



So $|MEP\rangle = \sqrt{\tfrac{1}{2}}\left(|0\rangle\otimes|0\rangle + |1\rangle\otimes|1\rangle\right)$

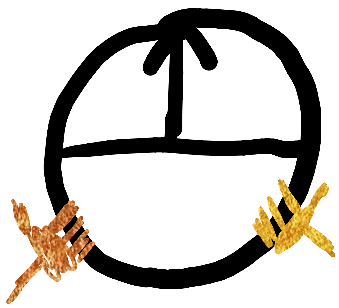
But: Alice does U on A $\rightsquigarrow$ $\sqrt{\tfrac{1}{2}}\left(|v\rangle\otimes|0\rangle + |v^\perp\rangle\otimes|0\rangle\right)$

then Bob does U on B $\rightsquigarrow$ $\underbrace{\sqrt{\tfrac{1}{2}}\left(|v\rangle\otimes|v\rangle + |v^\perp\rangle\otimes|v^\perp\rangle\right)}_{\text{THIS IS ALSO } |MEP\rangle\,!!}$

[Might seem weird, but it's true. Can't visualize 4-d vectors so easily, but <u>something</u> like:

any rot of 2 red vecs around green circle still adds up to blue...]

[We may also be interested in the joint state where we start with $|MEP\rangle$ and have some "different amount of rotation on the two halves"]



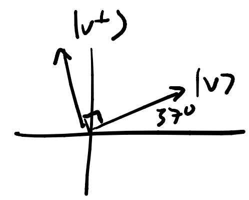
Alice does $Rot_{37°}$ on A $\longrightarrow$

$|MEP\rangle =$

$$\sqrt{\tfrac{1}{2}}\left(|0\rangle\otimes|0\rangle + |1\rangle\otimes|1\rangle\right)$$

$$\sqrt{\tfrac{1}{2}}\left(|v\rangle\otimes|0\rangle + |v^{\perp}\rangle\otimes|1\rangle\right)$$

where

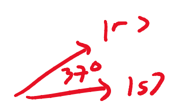


Could call this "$|MEP\ 37°\ tow.\ A\rangle$", though such notation is not standard!

[but also the same with

$|r\rangle\otimes|s\rangle + |r^{\perp}\rangle\otimes|s^{\perp}\rangle$

for any two $|r\rangle, |s\rangle$ at angle 37°



]

Could also get this state as,

e.g. :
- Start from $|MEP\rangle$
- Alice does $Rot_{100°}$ on A $\longrightarrow$ Equiv to Bob does $Rot_{-100°}$ on B
- Bob does $Rot_{63°}$ on B

Now equiv to $(Rot_{-37°}$ on $B)\,|MEP\rangle$

$\equiv (Rot_{37°}$ on $A)\,|MEP\rangle$

Measuring "$|\text{MEP rot } 37° \text{ tow. Alice}\rangle$":

([Recall, we computed this state is...])

$$\sqrt{\tfrac{1}{2}} \times \begin{array}{ccc} & 00 & 01 \\ & \begin{bmatrix} .8 & -.6 \\ .6 & .8 \end{bmatrix} & \\ & 10 & 11 \end{array} = \sqrt{\tfrac{1}{2}}\left(.8|\overset{AB}{00}\rangle - .6|\overset{AB}{01}\rangle + .6|\overset{AB}{10}\rangle + .8|\overset{AB}{11}\rangle\right)$$

$\updownarrow A \quad \leftarrow B \rightarrow$

If Alice measures A:

$$\Pr[\text{readout "0"}] = \left(\sqrt{\tfrac{1}{2}} \cdot .8\right)^2 + \left(\sqrt{\tfrac{1}{2}} \cdot (-.6)\right)^2 = \tfrac{1}{2}\left(.8^2 + (-.6)^2\right) = 1/2.$$

$\therefore$ "1" = $1/2$.

Same for Bob if he measures B!

$$\Pr[\text{readout "0"}] = \left(\sqrt{\tfrac{1}{2}} \cdot .8\right)^2 + \left(\sqrt{\tfrac{1}{2}} \cdot .6\right)^2 = \tfrac{1}{2}\left(.8^2 + .6^2\right) = 1/2$$

"1" $1/2$.

Same for $|\text{MEP rot } \boxed{\text{anything}} \text{ tow. Alice}\rangle$!
(or reflected)

[Above didn't depend on ".8", ".6". Would work for any $\cos\theta$, $\sin\theta$.]

* **Conclusion***:

Once Alice & Bob have $|\text{MEP}\rangle$, NOTHING they do <u>INDIVIDUALLY</u> to their bits can change the fact they see 50/50 "0"/"1" when measuring! [This can change if they do some <u>joint</u> ops on A & B.]

[Despite the above, MEPs are far from useless! We'll see many "magic tricks" you can do with MEPs...
In fact, we already saw one....]

Recall HW3.5 : "Super-dense coding":

1 MEP + sending 1 qubit $\Rightarrow$ sending 2 classical bits

Now: "Remote state preparation":

1 MEP + sending 1 classical bit $\Rightarrow$ sending 1 qubit

Say Alice & Bob get together [at Starbucks on Craig]

create $|MEP\rangle$,

Alice takes qubit A home [to CMU]

Bob " " B " [to Pitt]

Next, Alice dreams up some angle, $\theta = 67.7966 1016\ldots$ (say)

Her goal: get Bob to hold $Rot_\theta|0\rangle$. [Unentangled w/ anything]

[Note: it's trivial for Alice to get a qubit in this state. She knows θ, she can just do "Make A_2, Rot_θ on A_2" herself.]

Option 1: Text Bob all the digits of θ.

[But this could be any arbitrarily large # of classical bits to transmit.]

Idea 2: [The actual idea.]

Alice applies $Rot_{-\theta}$ to her half of MEP.

Equiv. to Bob applying Rot_θ to his half.

State now ["|MEP, angle θ tow. Bob>"]

$$\sqrt{\tfrac{1}{2}}\left(|0\rangle \otimes Rot_\theta |0\rangle + |1\rangle \otimes \underbrace{Rot_\theta |1\rangle}_{\text{aka } Rot_{\theta+90^\circ}|0\rangle} \right)$$

[State is still entangled; gotta change that.]

Now she measures.

$Pr[\text{readout "0"}] = \frac{1}{2}$, as we know! [Squared ampl. on "piece" $\sqrt{\tfrac{1}{2}}|0\rangle \otimes Rot_\theta|0\rangle$ with $A=0$]

& state "collapses" to

$|0\rangle \otimes Rot_\theta |0\rangle$ → unentangled, & Bob's half is desired state $Rot_\theta|0\rangle$! :)

$Pr[\text{readout "1"}] = 1/2$

& state "collapses" to

$|1\rangle \otimes Rot_\theta |1\rangle$ → unentangled, but Bob's state is $Rot_\theta |1\rangle = Rot_{\theta+90^\circ}|0\rangle$.

[Not quite right...]

Solution: Alice texts Bob her readout [= 1 classical bit]

If Bob gets "0", does nothing. If "1", applies Rot_{-90° to his B. Now <u>always</u> has $Rot_\theta|0\rangle$! :)