Machine Learning for Signal Processing

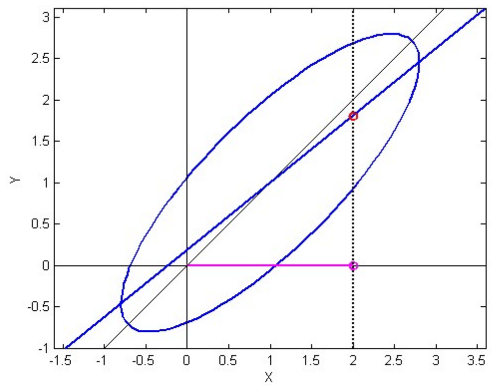
Predicting and Estimation from Time Series: Part 2

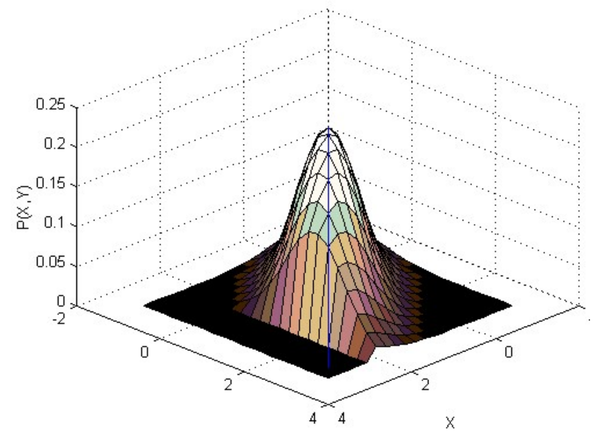
Bhiksha Raj

Preliminaries : P(y|x) for Gaussian

- If P(x,y) is Gaussian:

$$P(\mathbf{x},\mathbf{y}) = N\left(\begin{bmatrix}\mu_{\mathbf{x}} \\ \mu_{\mathbf{y}}\end{bmatrix}, \begin{bmatrix} C_{\mathbf{xx}} & C_{\mathbf{xy}} \\ C_{\mathbf{yx}} & C_{\mathbf{yy}}\end{bmatrix}\right)$$





- The conditional probability of y given x is also Gaussian
  - The slice in the figure is Gaussian

$$P(y\,|\,x) = N(\mu_y + C_{yx}C_{xx}^{-1}(x-\mu_x),\, C_{yy} - C_{yx}C_{xx}^{-1}C_{xy})$$

- The mean of this Gaussian is a function of x
- The variance of y reduces if x is known
  - Uncertainty is reduced

Background: Sum of Gaussian RVs

$$O = AS + \varepsilon$$

$$S \sim N(\mu_S, \Theta_S) \qquad \varepsilon \sim N(\mu_\varepsilon, \Theta_\varepsilon)$$

- The conditional probability of *O*:

$$P(O|S) = N(AS + \mu_\varepsilon, \Theta_\varepsilon)$$

- The overall probability of *O*:

$$P(O) = N(A\mu_s + \mu_\varepsilon, A\Theta_S A^T + \Theta_\varepsilon)$$

Background: Joint Prob. of O and S

$$O = AS + \varepsilon \qquad Z = \begin{bmatrix} O \\ S \end{bmatrix}$$

- The joint probability of *O* and *S* (i.e. P(*Z*)) is also Gaussian

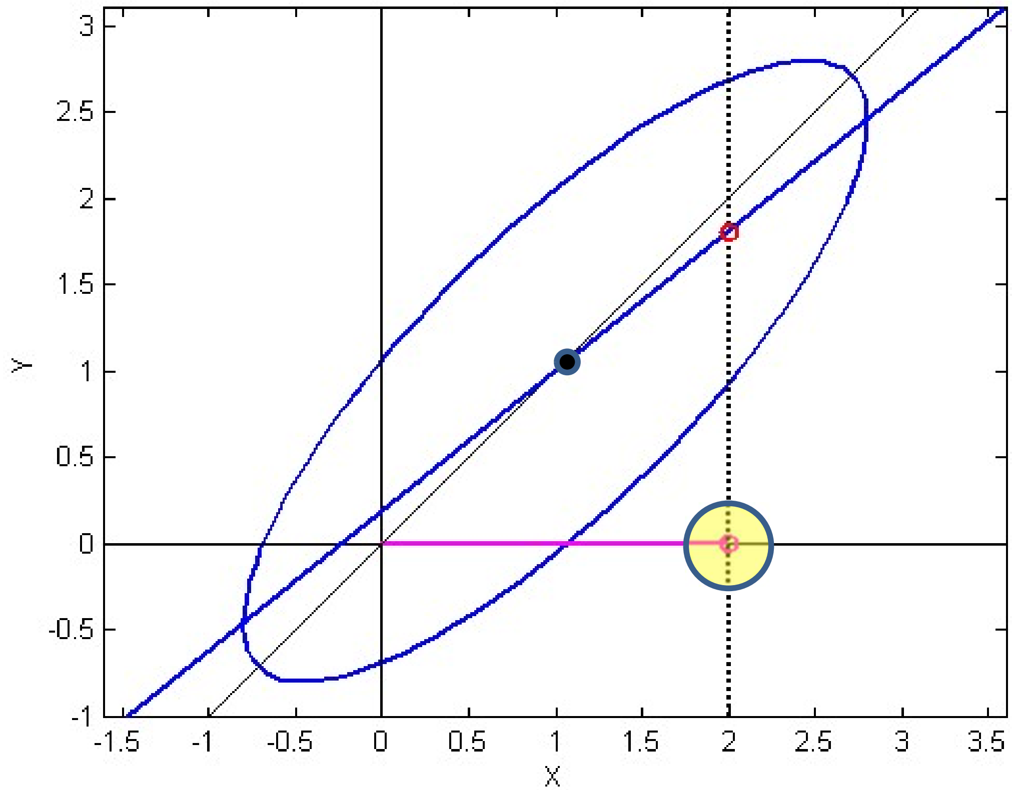
$$P(Z) = P(O, S) = N(\mu_Z, \Theta_Z)$$

- Where

$$\mu_Z = \begin{bmatrix} \mu_O \\ \mu_S \end{bmatrix} = \begin{bmatrix} A\mu_s + \mu_\varepsilon \\ \mu_S \end{bmatrix}$$

- $\Theta_Z = \begin{bmatrix} \Theta_O & \Theta_{OS} \\ \Theta_{SO} & \Theta_S \end{bmatrix} = \begin{bmatrix} A\Theta_S A^T + \Theta_\varepsilon & A\Theta_S \\ \Theta_S A^T & \Theta_S \end{bmatrix}$

Preliminaries : Conditional of S given O: P(S|O)

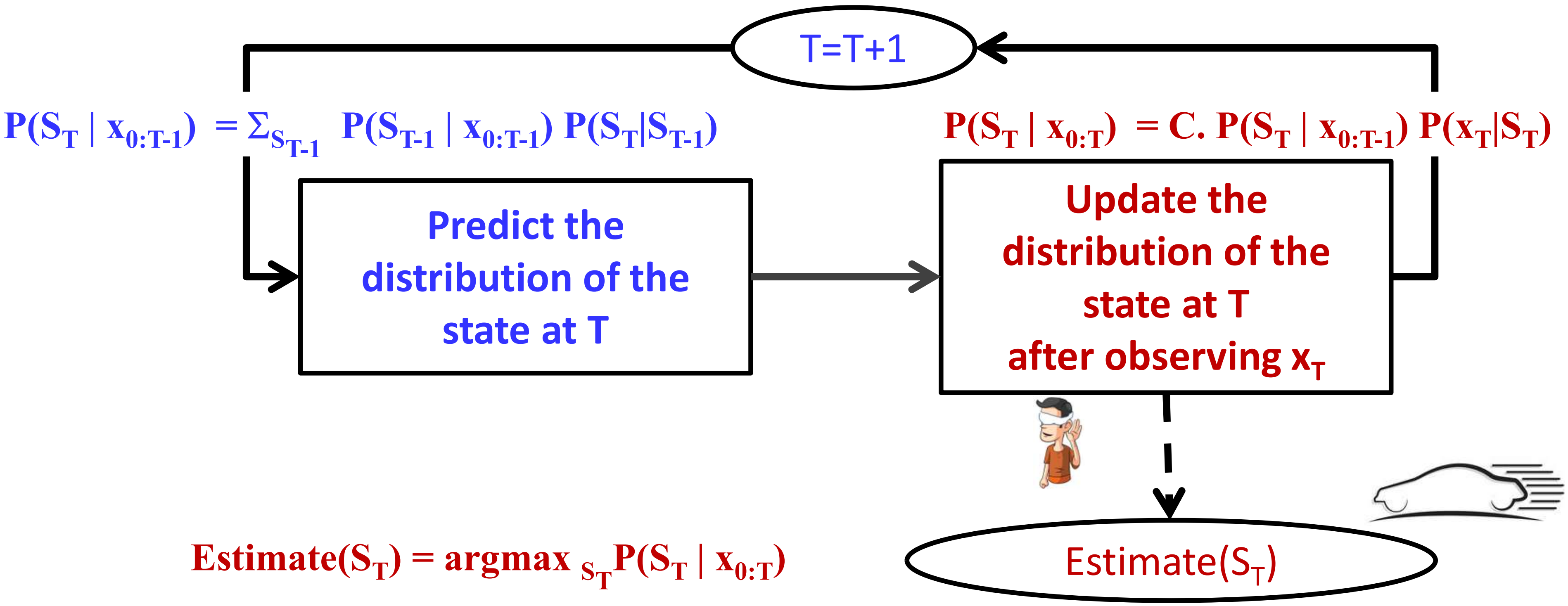


$$O = AS + \varepsilon$$

$$P(S|O) = N(\mu_S + \Theta_{SO}\Theta_O^{-1}(O - \mu_O),\ \ \Theta_S - \Theta_{SO}\Theta_O^{-1}\Theta_{OS})$$

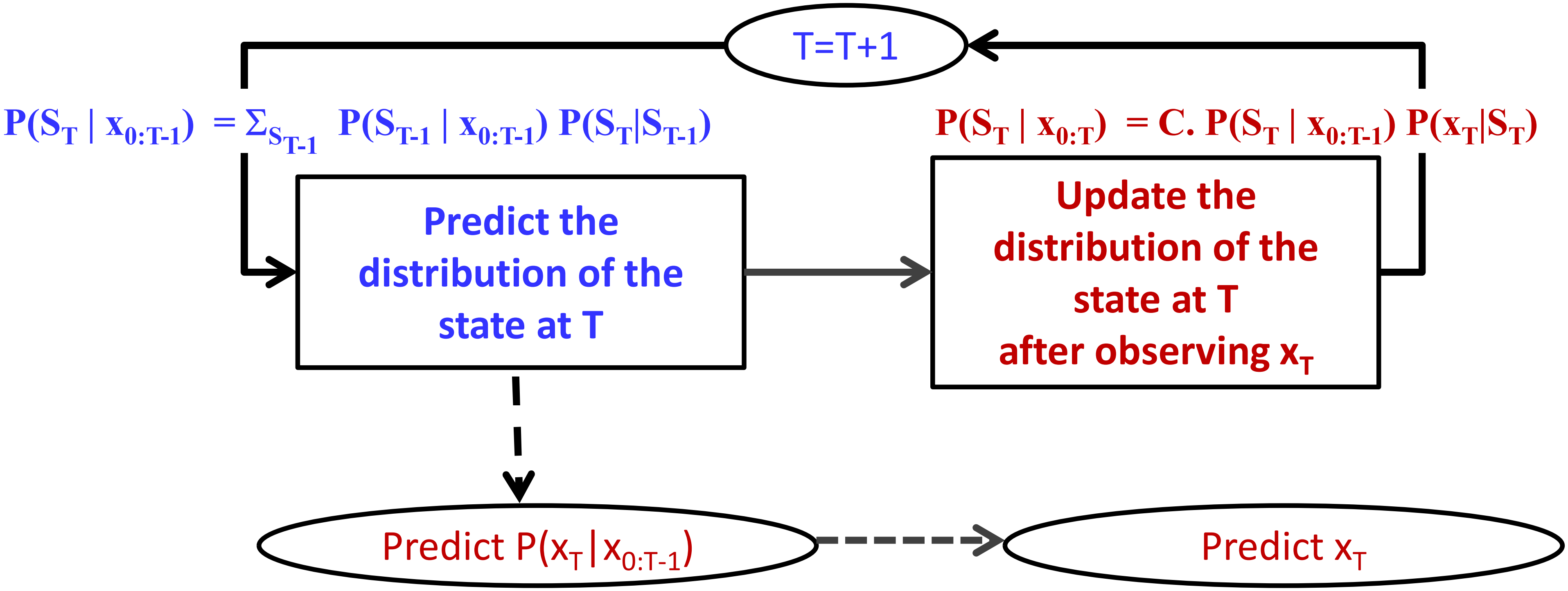
$$P(S|O) = N(\mu_S + \Theta_S A^{\mathrm{T}}\left(A\Theta_S A^{\mathrm{T}} + \Theta_\varepsilon\right)^{-1}(O - A\mu_s - \mu_\varepsilon),$$
$$\Theta_S - \Theta_S A^{\mathrm{T}}\left(A\Theta_S A^{\mathrm{T}} + \Theta_\varepsilon\right)^{-1} A\Theta_S)$$

Estimating the *state*



- The state is estimated from the updated distribution
  - The updated distribution is propagated into time, not the state

Predicting the *next observation*



- The probability distribution for the observations at the next time is a mixture:
- $P(X_t|X_{0:t-1}) = \sum_{S_t} P(X_t|S_t)P(S_t|X_{0:t-1})$
- The actual observation can be predicted from $P(x_T|x_{0:T-1})$

Predicting the next observation

- Can use any of the various estimators of x_T from $P(x_T|x_{0:T-1})$
- MAP estimate:
  - $\text{argmax}_{x_T}\ P(x_T|x_{0:T-1})$
- MMSE estimate:
  - $\text{Expectation}(x_T|x_{0:T-1})$

Difference from Viterbi decoding

- Estimating only the *current* state at any time
  - Not the state sequence
  - Although we are considering all past observations

- The most likely state at T and T+1 may be such that there is no valid transition between S_T and S_{T+1}

The real-valued state model

- A state equation describing the dynamics of the system

$$s_t = f(s_{t-1}, \varepsilon_t)$$

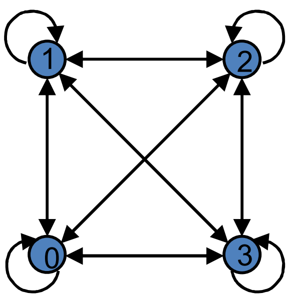
  - s_t is the state of the system at time t
  - ε_t is a driving function, which is assumed to be random
- The state of the system at any time depends only on the state at the previous time instant and the driving term at the current time

- An observation equation relating state to observation

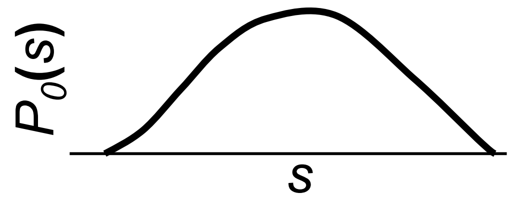
$$o_t = g(s_t, \gamma_t)$$

  - o_t is the observation at time t
  - γ_t is the noise affecting the observation (also random)
- The observation at any time depends only on the current state of the system and the noise

Discrete vs. Continuous state systems



$\pi =$



$$s_t = f(s_{t-1}, \varepsilon_t)$$

$$o_t = g(s_t, \gamma_t)$$

Prediction at time 0:

$$P(S_0) = \pi(S_0)$$

$$P(S_0) = P_0(S_0)$$

Update after O_0:

$$P(S_0|O_0) = C.\pi(S_0)P(O_0|S_0)$$

$$P(S_0|O_0) = C.P(S_0)P(O_0|S_0)$$

Prediction at time 1:

$$P(S_1|O_0) = \sum_{S_0} P(S_0|O_0)P(S_1|S_0)$$

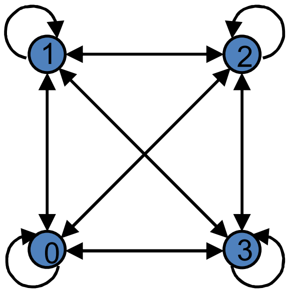
$$P(S_1|O_0) = \int_{-\infty}^{\infty} P(S_0|O_0)P(S_1|S_0)dS_0$$

Update after O_1:

$$P(S_1|O_{0:1}) = C.P(S_1|O_0)P(O_1|S_1)$$

$$P(S_1|O_{0:1}) = C.P(S_1|O_0)P(O_1|S_1)$$

Discrete vs. Continuous State Systems



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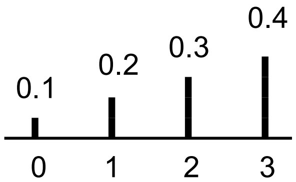


$$s_t = f(s_{t-1}, \varepsilon_t)$$

$$o_t = g(s_t, \gamma_t)$$

Prediction at time t:

$$P(S_t|O_{0:t-1}) = \sum_{S_{t-1}} P(S_{t-1}|O_{0:t-1})P(S_t|S_{t-1})$$

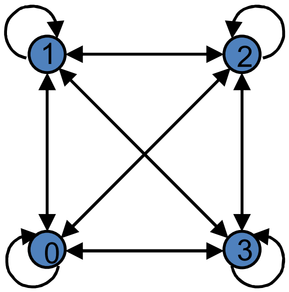
$$P(S_t|O_{0:t-1}) = \int_{-\infty}^{\infty} P(S_{t-1}|O_{0:t-1})P(S_t|S_{t-1})dS_{t-1}$$

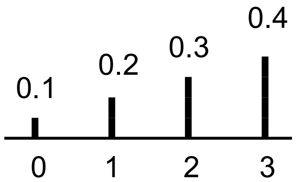
Update after observing O_t:

$$P(S_t|O_{0:t}) = C.P(S_t|O_{0:t-1})P(O_t|S_t)$$

$$P(S_t|O_{0:t}) = C.P(S_t|O_{0:t-1})P(O_t|S_t)$$

Discrete vs. Continuous State Systems



$\pi =$ 

$$s_t = f(s_{t-1}, \varepsilon_t)$$

$$o_t = g(s_t, \gamma_t)$$

Parameters

Initial state prob.	π	$P(s)$
Transition prob	$P(s_t = j \mid s_{t-1} = i)$	$P(s_t \mid s_{t-1})$
Observation prob	$P(O \mid s)$	$P(O \mid s)$

Special case: Linear Gaussian model

$$s_t = A_t s_{t-1} + \varepsilon_t$$

$$o_t = B_t s_t + \gamma_t$$

$$P(\varepsilon) = \frac{1}{\sqrt{(2\pi)^d \, |\Theta_\varepsilon|}} \exp\left(-0.5(\varepsilon - \mu_\varepsilon)^T \Theta_\varepsilon^{-1} (\varepsilon - \mu_\varepsilon)\right)$$

$$P(\gamma) = \frac{1}{\sqrt{(2\pi)^d \, |\Theta_\gamma|}} \exp\left(-0.5(\gamma - \mu_\gamma)^T \Theta_\gamma^{-1} (\gamma - \mu_\gamma)\right)$$

- A *linear* state dynamics equation
  - Probability of state driving term ε is Gaussian
  - Sometimes viewed as a driving term μ_ε and additive zero-mean noise
- A *linear* observation equation
  - Probability of observation noise γ is Gaussian
- A_t, B_t and Gaussian parameters assumed known
  - May vary with time

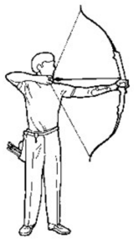
Linear model example
The wind and the target

- **State:** Wind speed at time t depends on speed at time t-1

$$S_t = S_{t-1} + \epsilon_t$$

- **Observation:** Arrow position at time *t* depends on wind speed at time *t*

$$O_t = BS_t + \gamma_t$$



Model Parameters: The initial state probability

$$P_0(s) = \frac{1}{\sqrt{(2\pi)^d \, |R|}} \exp\left(-0.5(s-\bar{s})R^{-1}(s-\bar{s})^T\right)$$

$$P_0(s) = Gaussian(s; \bar{s}, R)$$

- We also assume the *initial* state distribution to be Gaussian
  - Often assumed zero mean

$$s_t = A_t s_{t-1} + \varepsilon_t$$

$$o_t = B_t s_t + \gamma_t$$

Model Parameters: The observation probability

$$o_t = B_t s_t + \gamma_t$$

$$P(\gamma) = Gaussian(\gamma; \mu_\gamma, \Theta_\gamma)$$

$$P(o_t \mid s_t) = Gaussian(o_t; \mu_\gamma + B_t s_t, \Theta_\gamma)$$

- The probability of the observation, given the state, is simply the probability of the noise, with the mean shifted
  - Since the only uncertainty is from the noise
- The new mean is the mean of the distribution of the noise + the value of the observation in the absence of noise

Model Parameters: State transition probability

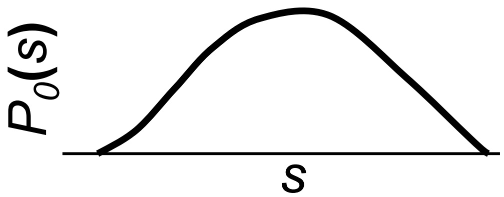
$$s_{t+1} = A_t s_t + \varepsilon_t$$

$$P(\varepsilon) = Gaussian(\varepsilon; \mu_\varepsilon, \Theta_\varepsilon)$$

$$P(s_{t+1} \mid s_t) = Gaussian(s_t; \mu_\varepsilon + A_t s_t, \Theta_\varepsilon)$$

- The probability of the state at time t, given the state at t-1, is simply the probability of the driving term, with the mean shifted

Continuous state systems



$$s_{t+1} = A_t s_t + \varepsilon_t$$

$$o_t = B_t s_t + \gamma_t$$

Prediction at time 0:

$$P(S_0) = P_0(S_0)$$

Update after O_0:

$$P(S_0|O_0) = C.P(S_0)P(O_0|S_0)$$

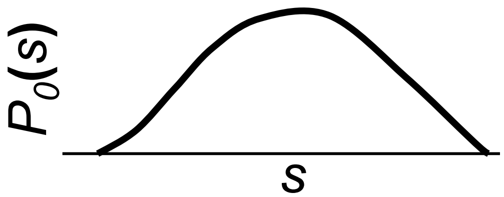
Prediction at time 1:

$$P(S_1|O_0) = \int_{-\infty}^{\infty} P(S_0|O_0)P(S_1|S_0)dS_0$$

Update after O_1:

$$P(S_1|O_{0:1}) = C.P(S_1|O_0)P(O_1|S_1)$$

Continuous state systems



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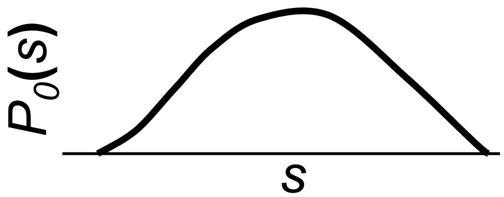
$$P_0(s) = Gaussian(s; \bar{s}_0, R_0)$$

- We assume the *initial* state distribution to be Gaussian
  - Often assumed zero mean

Continuous state systems



$$s_{t+1} = A_t s_t + \varepsilon_t$$

$$o_t = B_t s_t + \gamma_t$$

a priori probability distribution of state s

Prediction at time 0:

$$P(S_0) = P_0(S_0) \qquad = N(\bar{s}_0, R_0)$$

Update after O_0:

$$P(S_0|O_0) = C.P(S_0)P(O_0|S_0)$$

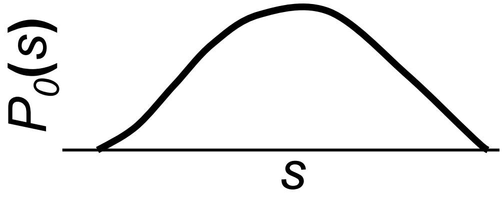
Prediction at time 1:

$$P(S_1|O_0) = \int_{-\infty}^{\infty} P(S_0|O_0)P(S_1|S_0)dS_0$$

Update after O_1:

$$P(S_1|O_{0:1}) = C.P(S_1|O_0)P(O_1|S_1)$$

Continuous state systems



$$s_{t+1} = A_t s_t + \varepsilon_t$$

$$o_t = B_t s_t + \gamma_t$$

Prediction at time 0:

$$\boldsymbol{P(S_0) = N(\bar{s}_0, R_0)}$$

Update after O_0:

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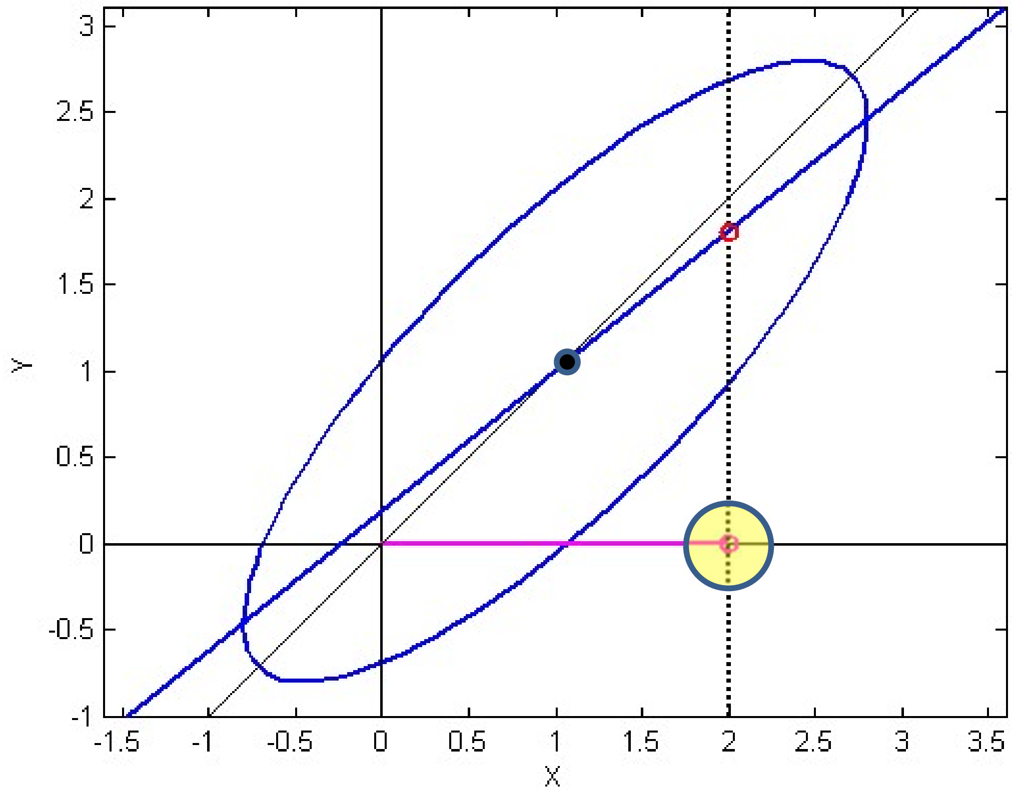
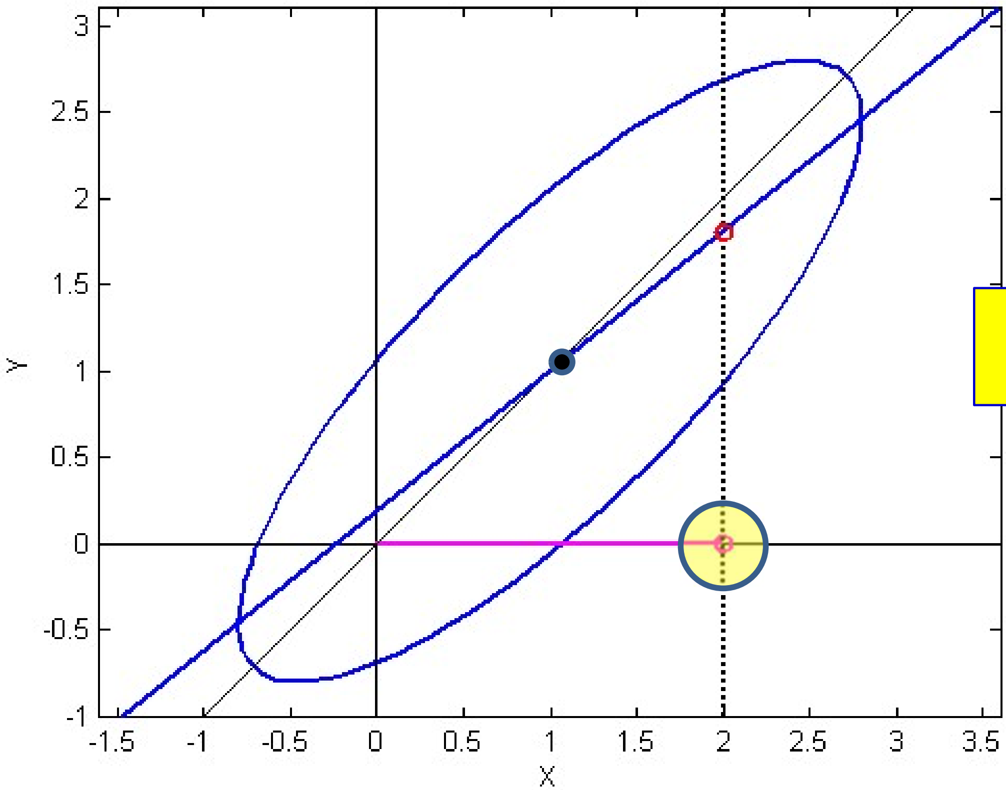
Recap: Conditional of S given O: P(S|O) for Gaussian RVs



$$O = BS + \gamma$$

$$P(S|O) = N(\mu_S + \Theta_{SO}\Theta_O^{-1}(O - \mu_O),\ \ \Theta_S - \Theta_{SO}\Theta_O^{-1}\Theta_{OS})$$

Recap: Conditional of S given O: P(S|O) for Gaussian RVs



$$O = BS + \gamma$$

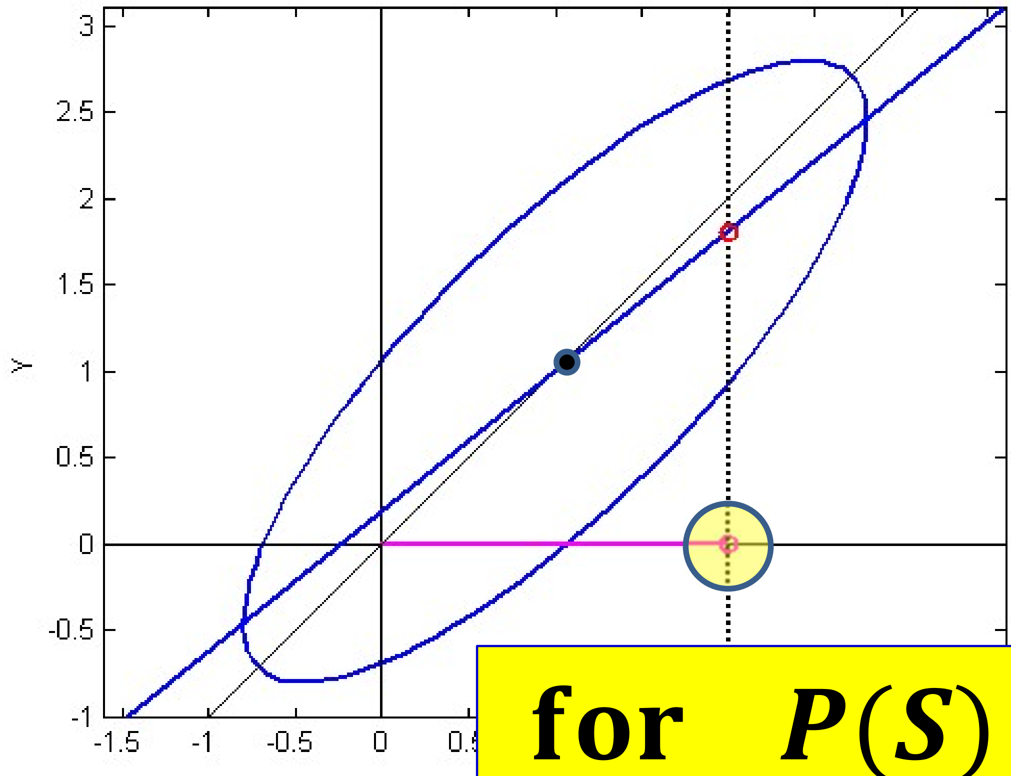
$$\Theta_{SO} = \Theta_S B^{\mathrm{T}}$$

$$\Theta_O = B\Theta_S B^{\mathrm{T}} + \Theta_\gamma$$

$$P(S|O) = N(\mu_S + \Theta_{SO}\Theta_O^{-1}(O - \mu_O),\ \ \Theta_S - \Theta_{SO}\Theta_O^{-1}\Theta_{OS})$$

$$P(S|O) = N(\mu_S + \Theta_S B^{\mathrm{T}}\left(B\Theta_S B^{\mathrm{T}} + \Theta_\gamma\right)^{-1}(O - B\mu_s - \mu_\gamma),$$
$$\Theta_S - \Theta_S B^{\mathrm{T}}\left(B\Theta_S B^{\mathrm{T}} + \Theta_\gamma\right)^{-1} B\Theta_S)$$

Recap: Conditional of S given O: P(S|O) for Gaussian RVs

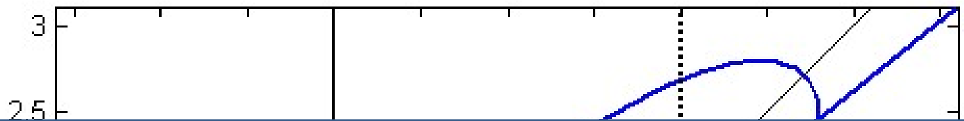


$$O = BS + \gamma$$

for $P(S) = N(\bar{s}_0, R_0)$

$$P(S_0|O_0) = N(\bar{s}_0 + R_0B^{\mathrm{T}}\left(BR_0B^{\mathrm{T}} + \Theta_\gamma\right)^{-1}(O_0 - B\bar{s}_0 - \mu_\gamma),$$
$$R_0 - R_0B^{\mathrm{T}}\left(BR_0B^{\mathrm{T}} + \Theta_\gamma\right)^{-1}BR_0)$$

Recap: Conditional of S given O: P(S|O) for Gaussian RVs



$$O = BS + \gamma$$

$$K_0 = R_0 B^{\mathrm{T}} \left(B R_0 B^{\mathrm{T}} + \Theta_\gamma \right)^{-1}$$

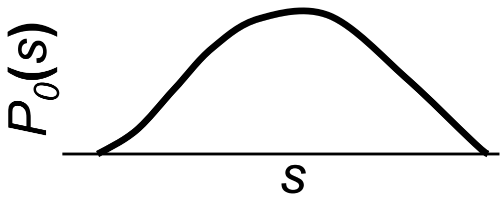
$$\hat{s}_0 = \bar{s}_0 + K_0 (O_0 - B\bar{s}_0 - \mu_\gamma)$$

$$\hat{R}_0 = (I - K_0)\, R_0$$

$$P(S_0|O_0) = N(\hat{s}_0, \hat{R}_0)$$

$$P(S_0|O_0) = N(\bar{s}_0 + R_0 B^{\mathrm{T}} \left(B R_0 B^{\mathrm{T}} + \Theta_\gamma \right)^{-1} (O_0 - B\bar{s}_0 - \mu_\gamma),$$
$$R_0 - R_0 B^{\mathrm{T}} \left(B R_0 B^{\mathrm{T}} + \Theta_\gamma \right)^{-1} B R_0)$$

Continuous state systems



$$s_{t+1} = A_t s_t + \varepsilon_t$$

$$o_t = B_t s_t + \gamma_t$$

Prediction at time 0:

$$\boldsymbol{P(S_0) = N(\bar{s}_0, R_0)}$$

Update after O_0:

$$P(S_0|O_0) = C.P(S_0)P(O_0|S_0)$$

$$P(S_0|O_0) = N(\hat{s}_0, \hat{R}_0)$$

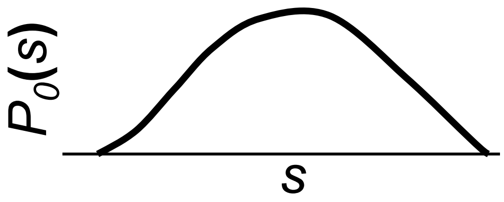
Prediction at time 1:

$$P(S_1|O_0) = \int_{-\infty}^{\infty} P(S_0|O_0)P(S_1|S_0)dS_0$$

Update after O_1:

$$P(S_1|O_{0:1}) = C.P(S_1|O_0)P(O_1|S_1)$$

Continuous state systems



$$s_{t+1} = A_t s_t + \varepsilon_t$$

$$o_t = B_t s_t + \gamma_t$$

Prediction at time 0:

$$P(S_0) = N(\bar{s}_0, R_0)$$

Update after O_0:

$$K_0 = R_0 B^{\mathrm{T}} \left(B R_0 B^{\mathrm{T}} + \Theta_\gamma \right)^{-1}$$

$$P(S_0|O_0) = N(\hat{s}_0, \hat{R}_0)$$

$$\hat{s}_0 = \bar{s}_0 + K_0(O_0 - B\bar{s}_0 - \mu_\gamma)$$

$$\hat{R}_0 = (I - K_0)\, R_0$$

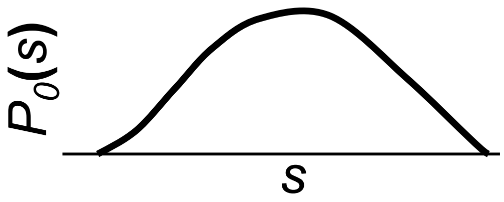
Prediction at time 1:

$$P(S_1|O_0) = \int_{-\infty}^{\infty} P(S_0|O_0) P(S_1|S_0) dS_0$$

Update after O_1:

$$P(S_1|O_{0:1}) = C.\, P(S_1|O_0) P(O_1|S_1)$$

Continuous state systems



$$s_{t+1} = A_t s_t + \varepsilon_t$$

$$o_t = B_t s_t + \gamma_t$$

Prediction at time 0:

$$\boldsymbol{P(S_0) = N(\bar{s}_0, R_0)}$$

Update after O_0:

$$P(S_0|O_0) = C.P(S_0)P(O_0|S_0)$$

$$= N(\bar{s}_0 + R_0 B^{\mathrm{T}}\left(BR_0 B^{\mathrm{T}} + \Theta_\gamma\right)^{-1}(O_0 - B\bar{s}_0 - \mu_\gamma),$$
$$R_0 - R_0 B^{\mathrm{T}}\left(BR_0 B^{\mathrm{T}} + \Theta_\gamma\right)^{-1} BR_0)$$

Prediction at time 1:

$$P(S_1|O_0) = \int_{-\infty}^{\infty} P(S_0|O_0)P(S_1|S_0)dS_0$$

Update after O_1:

$$P(S_1|O_{0:1}) = C.P(S_1|O_0)P(O_1|S_1)$$

Introducting shorthand notation

$$P(S_0|O_0) = N(\bar{s}_0 + R_0B^{\mathrm{T}}\left(BR_0B^{\mathrm{T}} + \Theta_\gamma\right)^{-1}(O_0 - B\bar{s}_0 - \mu_\gamma),$$
$$R_0 - R_0B^{\mathrm{T}}\left(BR_0B^{\mathrm{T}} + \Theta_\gamma\right)^{-1}BR_0)$$

$$\hat{s}_0 = \bar{s}_0 + R_0B^{\mathrm{T}}\left(BR_0B^{\mathrm{T}} + \Theta_\gamma\right)^{-1}(O - B\bar{s}_0 - \mu_\gamma)$$

$$\hat{R}_0 = R_0 - R_0B^{\mathrm{T}}\left(BR_0B^{\mathrm{T}} + \Theta_\gamma\right)^{-1}BR_0$$

$$P(S_0|O_0) = N(\hat{s}_0, \hat{R}_0)$$

Introducting shorthand notation

$$P(S_0|O_0) = N(\bar{s}_0 + R_0 B^{\mathrm{T}}\left(BR_0B^{\mathrm{T}} + \Theta_\gamma\right)^{-1}(O_0 - B\bar{s}_0 - \mu_\gamma),$$
$$R_0 - R_0B^{\mathrm{T}}\left(BR_0B^{\mathrm{T}} + \Theta_\gamma\right)^{-1}BR_0)$$

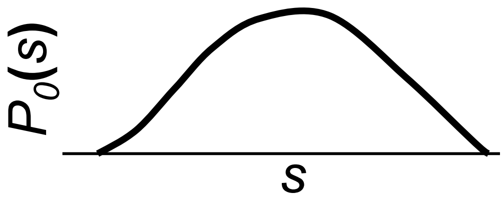
$$\boldsymbol{K_0 = R_0B^{\mathrm{T}}\left(BR_0B^{\mathrm{T}} + \Theta_\gamma\right)^{-1}}$$

$$\boldsymbol{\hat{s}_0 = \bar{s}_0 + K_0\,(O - B\bar{s}_0 - \mu_\gamma)}$$

$$\boldsymbol{\hat{R}_0 = (I - K_0B)R_0}$$

$$\boldsymbol{P(S_0|O_0) = N(\hat{s}_0, \hat{R}_0)}$$

Continuous state systems



$$s_{t+1} = A_t s_t + \varepsilon_t$$

$$o_t = B_t s_t + \gamma_t$$

Prediction at time 0:

$$\boldsymbol{P(S_0) = N(\bar{s}_0, R_0)}$$

Update after O_0:

$$P(S_0|O_0) = N(\hat{s}_0, \hat{R}_0)$$

$$\boldsymbol{K_0 = R_0 B^{\mathrm{T}} \left(B R_0 B^{\mathrm{T}} + \Theta_\gamma\right)^{-1}}$$

$$\boldsymbol{\hat{s}_0 = \bar{s}_0 + K_0(O_0 - B\bar{s}_0 - \mu_\gamma)}$$

$$\boldsymbol{\hat{R}_0 = (I - K_0)\, R_0}$$

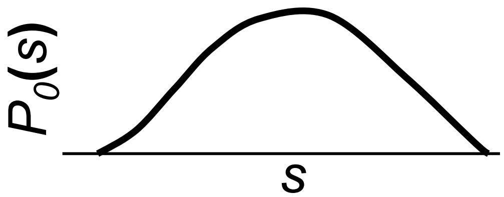
Prediction at time 1:

$$P(S_1|O_0) = \int_{-\infty}^{\infty} P(S_0|O_0) P(S_1|S_0) dS_0$$

Update after O_1:

$$P(S_1|O_{0:1}) = C.P(S_1|O_0)P(O_1|S_1)$$

Continuous state systems



$$s_{t+1} = A_t s_t + \varepsilon_t$$

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Prediction at time 0:

$$\boldsymbol{P(S_0) = N(\bar{s}_0, R_0)}$$

Update after O_0:

$$K_0 = R_0 B^{\mathrm{T}} \left(B R_0 B^{\mathrm{T}} + \Theta_\gamma\right)^{-1}$$

$$P(S_0|O_0) = N(\hat{s}_0, \hat{R}_0)$$

$$\hat{s}_0 = \bar{s}_0 + K_0(O_0 - B\bar{s}_0 - \mu_\gamma)$$

$$\hat{R}_0 = (I - K_0)\, R_0$$

Prediction at time 1:

$$P(S_1|O_0) = \int_{-\infty}^{\infty} P(S_0|O_0) P(S_1|S_0) dS_0$$

Update after O_1:

$$P(S_1|O_{0:1}) = C.P(S_1|O_0)P(O_1|S_1)$$

The prediction equation

$$P(S_1|O_0) = \int_{-\infty}^{\infty} P(S_0|O_0)P(S_1|S_0)dS_0$$

$$P(S_0|O_0) = N(\hat{\mathbf{s}}_0, \hat{\mathbf{R}}_0)$$

$$P(S_1|S_0) = N(\mathbf{A}S_0 + \boldsymbol{\mu}_{\boldsymbol{\varepsilon}}, \boldsymbol{\Theta}_{\boldsymbol{\varepsilon}})$$

$$P(\varepsilon) = N(\mu_\varepsilon, \Theta_\varepsilon)$$

$$s_{t+1} = A_t s_t + \varepsilon_t$$

- The integral of the product of two Gaussians

$$P(S_1|O_0) = \int_{-\infty}^{\infty} Gaussian(S_0; \hat{s}_0, \hat{R}_0) Gaussian(S_1; AS_0, \Theta_\varepsilon)dS_0$$

The Prediction Equation

- The integral of the product of two Gaussians is Gaussian!

$$P(S_1|O_0) = \int_{-\infty}^{\infty} Gaussian\left(S_0; \hat{s}_0, \hat{R}_0\right) Gaussian(S_1; AS_0 + \mu_\varepsilon, \Theta_\varepsilon) dS_0$$

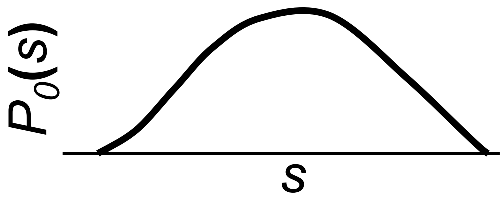
$$= \int_{-\infty}^{\infty} C_1 exp\left(-0.5(S_0 - \hat{s}_0)\hat{R}_0^{-1}(S_0 - \hat{s}_0)^T\right). C_2 exp(-0.5(S_1 - AS_0 - \mu_\varepsilon)\Theta_\varepsilon^{-1}(S_1 - AS_0 - \mu_\varepsilon)^T) dS_0$$

$$= Gaussian(S_1; A\hat{s}_0 + \mu_\varepsilon, \Theta_\varepsilon + A\hat{R}_0 A^T)$$

$$s_{t+1} = A_t s_t + \varepsilon_t$$

$$P(S_1|O_0) = N(A\hat{s}_0 + \mu_\varepsilon, \Theta_\varepsilon + A\hat{R}_0 A^T)$$

Continuous state systems



$$s_{t+1} = A_t s_t + \varepsilon_t$$

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Prediction at time 0:

$$\boldsymbol{P(S_0) = N(\bar{s}_0, R_0)}$$

Update after O_0:

$$P(S_0|O_0) = N(\hat{s}_0, \hat{R}_0)$$

$$\boldsymbol{K_0 = R_0 B^T (B R_0 B^T + \Theta_\gamma)^{-1}}$$

$$\boldsymbol{\hat{s}_0 = \bar{s}_0 + K_0(O_0 - B\bar{s}_0 - \mu_\gamma)}$$

$$\boldsymbol{\hat{R}_0 = (I - K_0)\, R_0}$$

Prediction at time 1:

$$P(S_1|O_0) = \int_{-\infty}^{\infty} P(S_0|O_0)P(S_1|S_0)dS_0$$

$$= N(A\hat{s}_0 + \mu_\varepsilon, \Theta_\varepsilon + A\hat{R}_0 A^T)$$

Update after O_1:

$$P(S_1|O_{0:1}) = C.P(S_1|O_0)P(O_1|S_1)$$

More shorthand notation

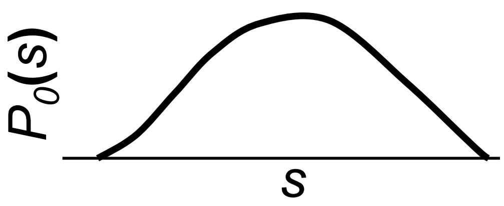
$$P(S_1|O_0) = N(A\hat{s}_0 + \mu_\varepsilon, \Theta_\varepsilon + A\hat{R}_0A^T)$$

$$\bar{s}_1 = A\hat{s}_0 + \mu_\varepsilon$$

$$R_1 = \Theta_\varepsilon + A\hat{R}_0A^T$$

$$P(S_1|O_0) = N(\bar{s}_1, R_1)$$

Continuous state systems



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Prediction at time 0:

$$P(S_0) = N(\bar{s}_0, R_0)$$

Update after O_0:

$$P(S_0|O_0) = N(\hat{s}_0, \hat{R}_0)$$

$$K_0 = R_0 B^{\mathrm{T}}\left(B R_0 B^{\mathrm{T}} + \Theta_\gamma\right)^{-1}$$

$$\hat{s}_0 = \bar{s}_0 + K_0(O_0 - B\bar{s}_0 - \mu_\gamma)$$

$$\hat{R}_0 = (I - K_0)\, R_0$$

Prediction at time 1:

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$$\bar{s}_1 = A\hat{s}_0 + \mu_\varepsilon$$

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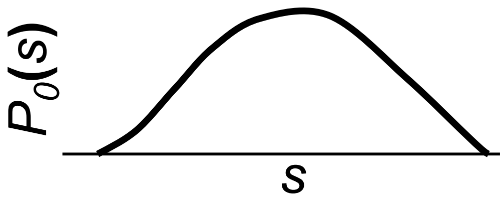
Continuous state systems



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Prediction at time 1:

$$P(S_1|O_0) = N(\bar{s}_1, R_1)$$

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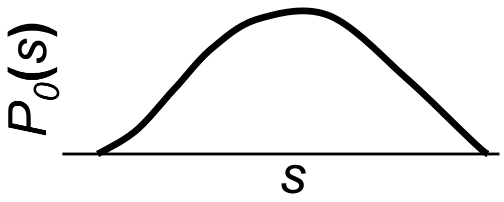
Continuous state systems



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$$P(S_0|O_0) = N(\hat{s}_0, \hat{R}_0)$$

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Prediction at time 1:

$$P(S_1|O_0) = N(\bar{s}_1, R_1)$$

$$\boldsymbol{\bar{s}_1 = A\hat{s}_0 + \mu_\varepsilon}$$

$$\boldsymbol{R_1 = \Theta_\varepsilon + A\hat{R}_0 A^T}$$

Update after O_1:

$$P(S_1|O_{0:1}) = C.P(S_1|O_0)P(O_1|S_1) = N(\hat{s}_1, \hat{R}_1)$$

$$\boldsymbol{K_1 = R_1 B^T (B R_1 B^T + \Theta_\gamma)^{-1}}$$

$$\boldsymbol{\hat{s}_1 = \bar{s}_1 + K_1 (O_1 - B\bar{s}_1 - \mu_\gamma)}$$

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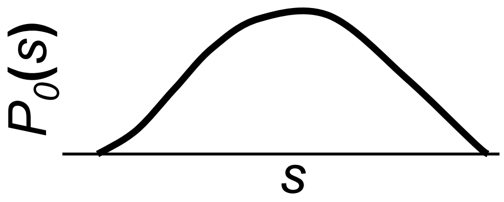
Continuous state systems



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$$P(S_0|O_0) = N(\hat{s}_0, \hat{R}_0)$$

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Update after O_1:

$$P(S_1|O_{0:1}) = N(\hat{s}_1, \hat{R}_1)$$

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$$\boldsymbol{\hat{s}_1 = \bar{s}_1 + K_1 (O_1 - B\bar{s}_1 - \mu_\gamma)}$$

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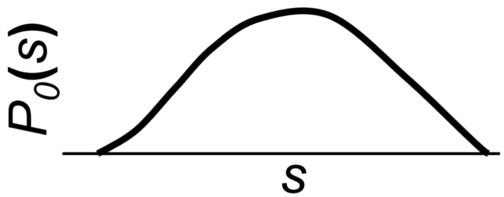
Gaussian Continuous State Linear Systems



$$s_{t+1} = A_t s_t + \varepsilon_t$$

$$o_t = B_t s_t + \gamma_t$$

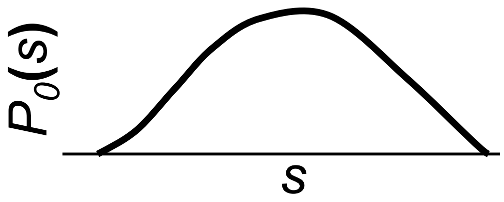
Prediction at time t:

$$P(S_t|O_{0:t-1}) = \int_{-\infty}^{\infty} P(S_{t-1}|O_{0:t-1})P(S_t|S_{t-1})dS_{t-1}$$

Update after observing O_t:

$$P(S_t|O_{0:t}) = C.P(S_t|O_{0:t-1})P(O_t|S_t)$$

Gaussian Continuous State Linear Systems



$$s_{t+1} = A_t s_t + \varepsilon_t$$

$$o_t = B_t s_t + \gamma_t$$

Prediction at time t:

$$P(S_t | O_{0:t-1}) = N(\bar{s}_t, R_t)$$

$$\bar{s}_t = A\hat{s}_{t-1} + \mu_\varepsilon$$

$$R_t = \Theta_\varepsilon + A\hat{R}_{t-1}A^T$$

Update after observing O_t:

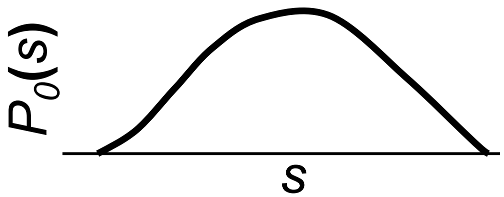
$$P(S_t | O_{0:t}) = N(\hat{s}_t, \hat{R}_t)$$

$$K_t = R_1 B^T \left(B R_1 B^T + \Theta_\gamma \right)^{-1}$$

$$\hat{s}_t = \bar{s}_t + Kt\,(Ot - B\bar{s}_t - \mu_\gamma)$$

$$\hat{R}_t = (I - KtB)\,R_t$$

Gaussian Continuous State Linear Systems



$$s_{t+1} = A_t s_t + \varepsilon_t$$

$$o_t = B_t s_t + \gamma_t$$

Prediction at time t:

$$P(S_t | O_{0:t-1}) = N(\bar{s}_t, R_t)$$

Update after observing O_t:

$$P(S_t | O_{0:t}) = N(\hat{s}_t, \hat{R}_t)$$

KALMAN FILTER

$$\bar{s}_t = A\hat{s}_{t-1} + \mu_\varepsilon$$

$$R_t = \Theta_\varepsilon + A\hat{R}_{t-1}A^T$$

$$K_t = R_1 B^T \left(B R_1 B^T + \Theta_\gamma\right)^{-1}$$

$$\hat{s}_t = \bar{s}_t + Kt\,(Ot - B\bar{s}_t - \mu_\gamma)$$

$$\hat{R}_t = (I - KtB)\,R_t$$

The Kalman filter

- Prediction (based on state equation)

$$\bar{s}_t = A_t \hat{s}_{t-1} + \mu_\varepsilon$$

$$s_t = A_t s_{t-1} + \varepsilon_t$$

$$R_t = \Theta_\varepsilon + A_t \hat{R}_{t-1} A_t^T$$

- Update (using observation and observation equation)

$$K_t = R_t B_t^T \left(B_t R_t B_t^T + \Theta_\gamma \right)^{-1}$$

$$o_t = B_t s_t + \gamma_t$$

$$\hat{s}_t = \bar{s}_t + K_t \left(o_t - B_t \bar{s}_t - \mu_\gamma \right)$$

$$\hat{R}_t = \left(I - K_t B_t \right) R_t$$

Explaining the Kalman Filter

- Prediction

$$s_t = A_t s_{t-1} + \varepsilon_t$$

$$\bar{s}_t = A_t \hat{s}_{t-1} + \mu_\varepsilon$$

$$o_t = B_t s_t + \gamma_t$$

$$R_t = \Theta_\varepsilon + A_t \hat{R}_{t-1} A_t^T$$

- The Kalman filter can be explained intuitively without working through the math

$$\hat{s}_t = \bar{s}_t + K_t \left(o_t - B_t \bar{s}_t - \mu_\gamma\right)$$

$$\hat{R}_t = \left(I - K_t B_t\right) R_t$$

Explaining the Kalman Filter

$$s_t = A_t s_{t-1} + \varepsilon_t$$

$$o_t = B_t s_t + \gamma_t$$

- Prediction

$$\bar{s}_t = A_t \hat{s}_{t-1} + \mu_\varepsilon$$

$$R_t = \Theta_\varepsilon + A_t \hat{R}_{t-1} A_t^T$$

- The Kalman filter can be explained intuitively without working through the math

$$\hat{s} = \bar{s} + K(o - B\bar{s} - \mu)$$

To do so, we must think of the filter as estimating (a) the state, and (b) the uncertainty of the estimate

Prediction

$$s_t = A_t s_{t-1} + \varepsilon_t$$

$$o_t = B_t s_t + \gamma_t$$

- If our best guess for the state at time $t-1$ is $\hat{s}_{t-1}$, what is our best prediction for s_t?

- If the guess $\hat{s}_{t-1}$ as uncertainty (variance) $\hat{R}_{t-1}$, what is the uncertainty of the prediction of the state at t?

The Kalman filter

- Prediction

$$s_t = A_t s_{t-1} + \varepsilon_t$$

$$o_t = B_t s_t + \gamma_t$$

$$\bar{s}_t = A_t \hat{s}_{t-1} + \mu_\varepsilon$$

The *predicted* state at time t is obtained simply by propagating the estimated state at t-1 through the state dynamics equation

$$K_t = R_t B_t^T \left(B_t R_t B_t^T + \Theta_\gamma\right)^{-1}$$

$$\hat{s}_t = \bar{s}_t + K_t\left(o_t - B_t \bar{s}_t - \mu_\gamma\right)$$

$$\hat{R}_t = \left(I - K_t B_t\right) R_t$$

The Kalman filter

- Prediction

$$\bar{s}_t = A_t \hat{s}_{t-1} + \mu_\varepsilon$$

$$s_t = A_t s_{t-1} + \varepsilon_t$$

$$o_t = B_t s_t + \gamma_t$$

$$R_t = \Theta_\varepsilon + A_t \hat{R}_{t-1} A_t^T$$

This is the uncertainty in the prediction. The variance of the predictor = variance of ε_t + variance of As_{t-1}

The two simply add because ε_t is not correlated with s_t

The Kalman filter

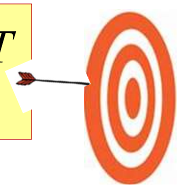
- Prediction

$$s_t = A_t s_{t-1} + \varepsilon_t$$

$$\bar{s}_t = A_t \hat{s}_{t-1} + \mu_\varepsilon$$

$$o_t = B_t s_t + \gamma_t$$

$$R_t = \Theta_\varepsilon + A_t \hat{R}_{t-1} A_t^T$$



$$\hat{o}_t = B_t \bar{s}_t + \mu_\gamma$$

We can also predict the *observation* from the predicted state using the observation equation

$$\hat{s}_t = \bar{s}_t + K_t(o_t - B_t \bar{s}_t - \mu_\gamma)$$

$$\hat{R}_t = (I - K_t B_t) R_t$$

Prediction

$$s_t = A_t s_{t-1} + \varepsilon_t$$

$$o_t = B_t s_t + \gamma_t$$

- If our best prediction for the state at time t is $\bar{s}_t$, what is our best prediction for o_t?

Prediction

$$s_t = A_t s_{t-1} + \varepsilon_t$$

$$o_t = B_t s_t + \gamma_t$$

- If our best prediction for the state at time t is $\bar{s}_t$, what is our best prediction for o_t?
  - If $\bar{s}_t$ has uncertainty (variance) R_t, what is the uncertainty of the prediction of the observation at t?

Prediction

$$s_t = A_t s_{t-1} + \varepsilon_t$$

$$o_t = B_t s_t + \gamma_t$$

- If our best prediction for the state at time t is $\bar{s}_t$, what is our best prediction for o_t?
  - If $\bar{s}_t$ has uncertainty (variance) R_t, what is the uncertainty of the prediction of the observation at t?

- Will the predicted $\hat{o}_t$ be the same as the actual observation of o_t?

Prediction

$$s_t = A_t s_{t-1} + \varepsilon_t$$

$$o_t = B_t s_t + \gamma_t$$

- If our best prediction for the state at time t is $\bar{s}_t$, what is our best prediction for o_t?
  - If $\bar{s}_t$ has uncertainty (variance) R_t, what is the uncertainty of the prediction of the observation at t?
- Will the predicted $\hat{o}_t$ be the same as the actual observation of o_t?
  - How should we adjust our guess $\bar{s}_t$ to account for this difference?

The Kalman filter

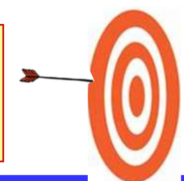
$$s_t = A_t s_{t-1} + \varepsilon_t$$

$$o_t = B_t s_t + \gamma_t$$

- Prediction

$$\bar{s}_t = A_t \hat{s}_{t-1} + \mu_\varepsilon$$

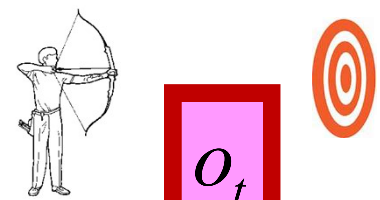
$$R_t = \Theta_\varepsilon + A_t \hat{R}_{t-1} A_t^T$$



$$\hat{o}_t = B_t \bar{s}_t + \mu_\gamma$$

- Update

Actual observation

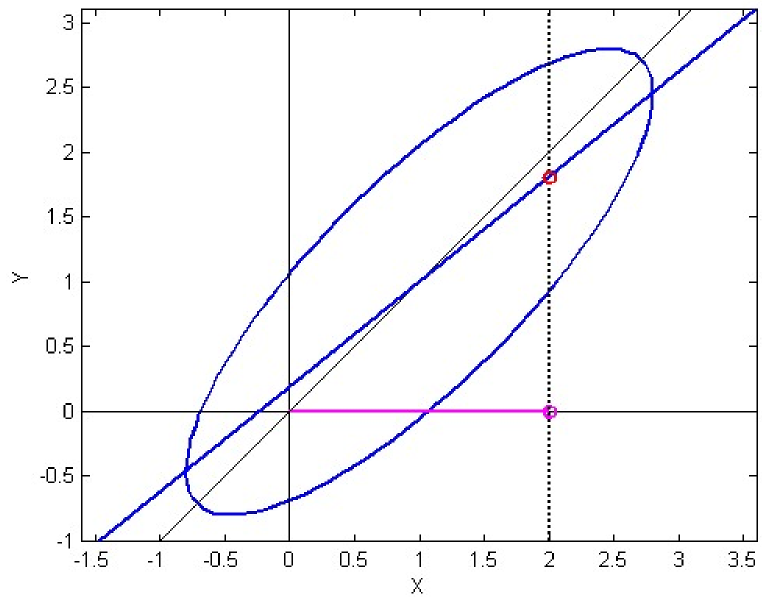


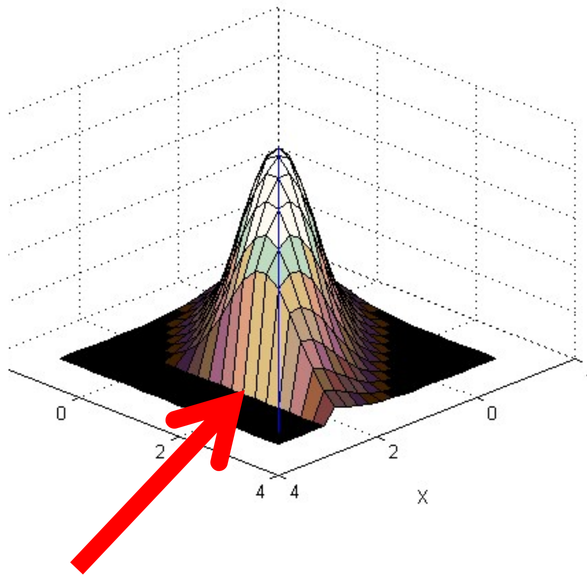
o_t

MAP Recap (for Gaussians)

- If P(x,y) is Gaussian:

$$P(\mathbf{x},\mathbf{y}) = N\left(\begin{bmatrix}\mu_{\mathbf{x}} \\ \mu_{\mathbf{y}}\end{bmatrix}, \begin{bmatrix} C_{\mathbf{xx}} & C_{\mathbf{xy}} \\ C_{\mathbf{yx}} & C_{\mathbf{yy}}\end{bmatrix}\right)$$





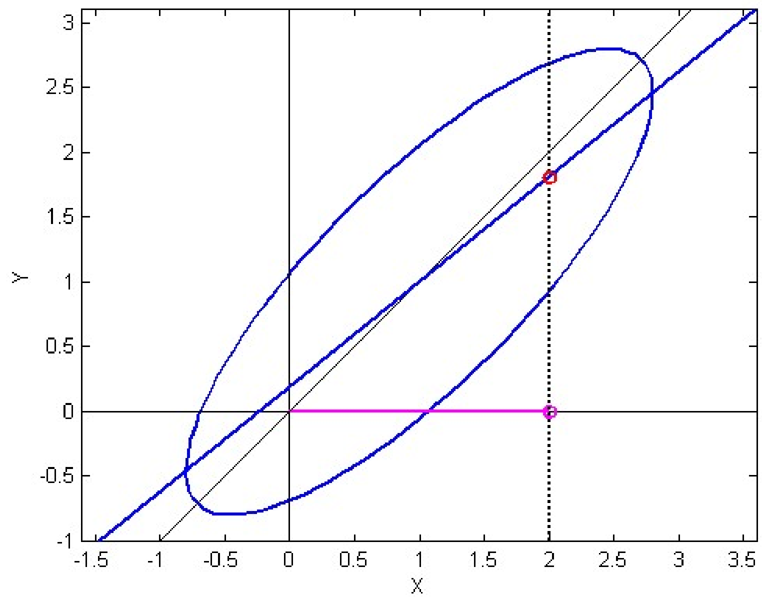
$$P(y \mid x) = N(\mu_y + C_{yx}C_{xx}^{-1}(x-\mu_x), C_{yy} - C_{yx}^T C_{xx}^{-1} C_{xy})$$

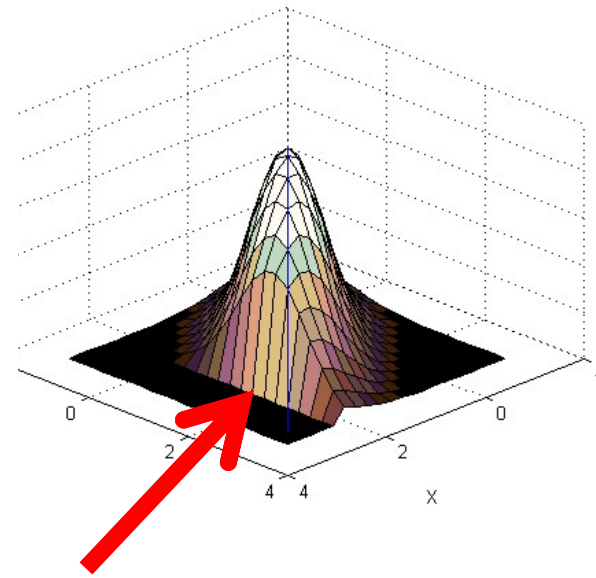
$$\hat{y} = \mu_y + C_{yx}C_{xx}^{-1}(x-\mu_x)$$

MAP Recap: For Gaussians

- If P(x,y) is Gaussian:

$$P(\mathbf{y},\mathbf{x}) = N\left(\begin{bmatrix}\mu_\mathbf{x}\\ \mu_\mathbf{y}\end{bmatrix}, \begin{bmatrix}C_\mathbf{xx} & C_\mathbf{xy}\\ C_\mathbf{yx} & C_\mathbf{yy}\end{bmatrix}\right)$$





$$P(y \mid x) = N(\mu_y + C_{yx}C_{xx}^{-1}(x-\mu_x), C_{yy} - C_{yx}^T C_{xx}^{-1} C_{xy})$$

$$\hat{y} = \mu_y + C_{yx}C_{xx}^{-1}(x-\mu_x)$$

"Slope" of the line

The Kalman filter

$$s_t = A_t s_{t-1} + \varepsilon_t$$

$$o_t = B_t s_t + \gamma_t$$

- Prediction

$$\bar{s}_t = A_t \hat{s}_{t-1} + \mu_\varepsilon$$

$$R_t = \Theta_\varepsilon + A_t \hat{R}_{t-1} A_t^T$$

$$\hat{o}_t = B_t \bar{s}_t + \mu_\gamma$$

- Update

$$o_t$$

$$K_t = R_t B_t^T \left(B_t R_t B_t^T + \Theta_\gamma \right)^{-1}$$

This is the slope of the MAP estimator that predicts s from o

$RB^T = C_{so}$, $(BRB^T + \Theta) = C_{oo}$

This is also called the **Kalman Gain**

The Kalman filter

$$s_t = A_t s_{t-1} + \varepsilon_t$$

- Prediction

$$\bar{s}_t = A_t \hat{s}_{t-1} + \mu_\varepsilon$$

We must correct the predicted value of the state after making an observation

$$\hat{o}_t = B_t \bar{s}_t + \mu_\gamma$$

$$o_t$$

$$K_t = R_t B_t^T \left(B_t R_t B_t^T + \Theta_\gamma \right)^{-1}$$

$$\hat{s}_t = \bar{s}_t + K_t \left(o_t - \hat{o}_t \right)$$

The correction is the difference between the *actual* observation and the *predicted* observation, scaled by the Kalman Gain

The Kalman filter

$$s_t = A_t s_{t-1} + \varepsilon_t$$

- Prediction

$$\bar{s}_t = A_t \hat{s}_{t-1} + \mu_\varepsilon$$

We must correct the predicted value of the state after making an observation

$$\hat{o}_t = B_t \bar{s}_t + \mu_\gamma$$

$$o_t$$

$$K_t = R_t B_t^T \left(B_t R_t B_t^T + \Theta_\gamma\right)^{-1}$$

$$\hat{s}_t = \bar{s}_t + K_t\left(o_t - B_t \bar{s}_t - \mu_\gamma\right)$$

The correction is the difference between the *actual* observation and the *predicted* observation, scaled by the Kalman Gain

The Kalman filter

$$s_t = A_t s_{t-1} + \varepsilon_t$$

$$o_t = B_t s_t + \gamma_t$$

- Prediction

$$\bar{s}_t = A_t \hat{s}_{t-1} + \mu_\varepsilon$$

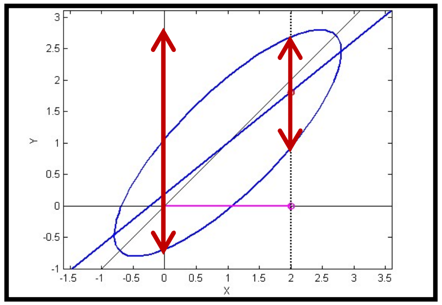
$$R_t = \Theta_\varepsilon + A_t \hat{R}_{t-1} A_t^T$$

- Update:

The uncertainty in state decreases if we observe the data and make a correction

The reduction is a multiplicative "shrinkage" based on Kalman gain and B

$$\hat{R}_t = (I - K_t B_t) R_t$$



The Kalman filter

$$s_t = A_t s_{t-1} + \varepsilon_t$$

$$o_t = B_t s_t + \gamma_t$$

- Prediction

$$\bar{s}_t = A_t \hat{s}_{t-1} + \mu_\varepsilon$$

$$R_t = \Theta_\varepsilon + A_t \hat{R}_{t-1} A_t^T$$

- Update:

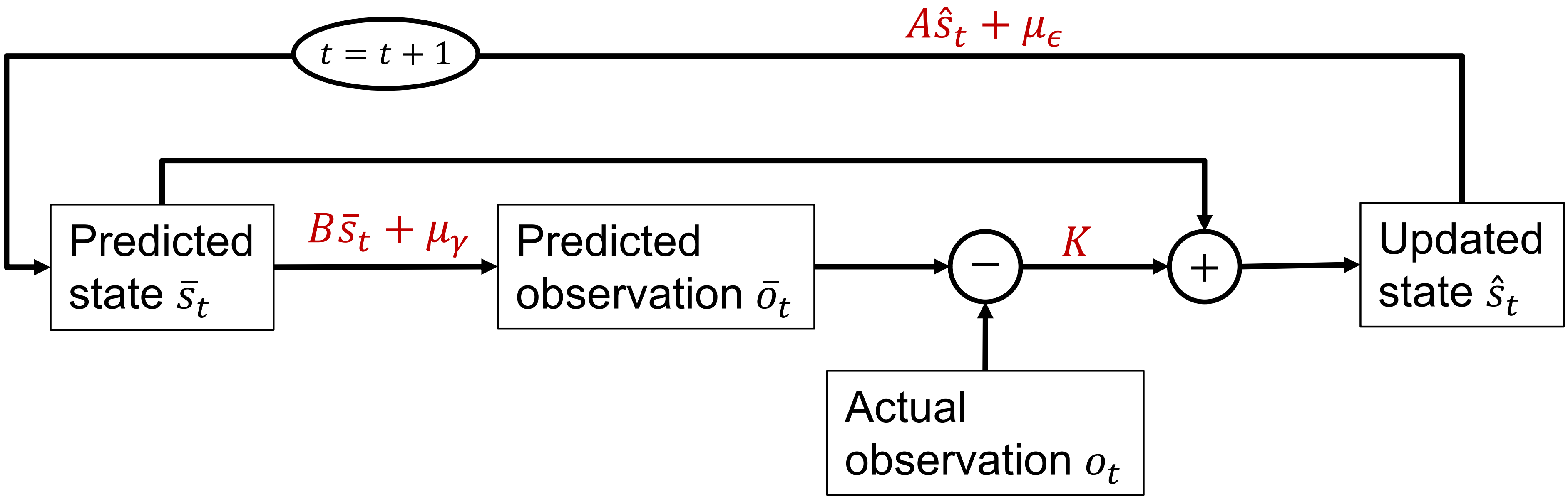
$$K_t = R_t B_t^T \left(B_t R_t B_t^T + \Theta_\gamma \right)^{-1}$$

- Update

$$\hat{s}_t = \bar{s}_t + K_t \left(o_t - B_t \bar{s}_t - \mu_\gamma \right)$$

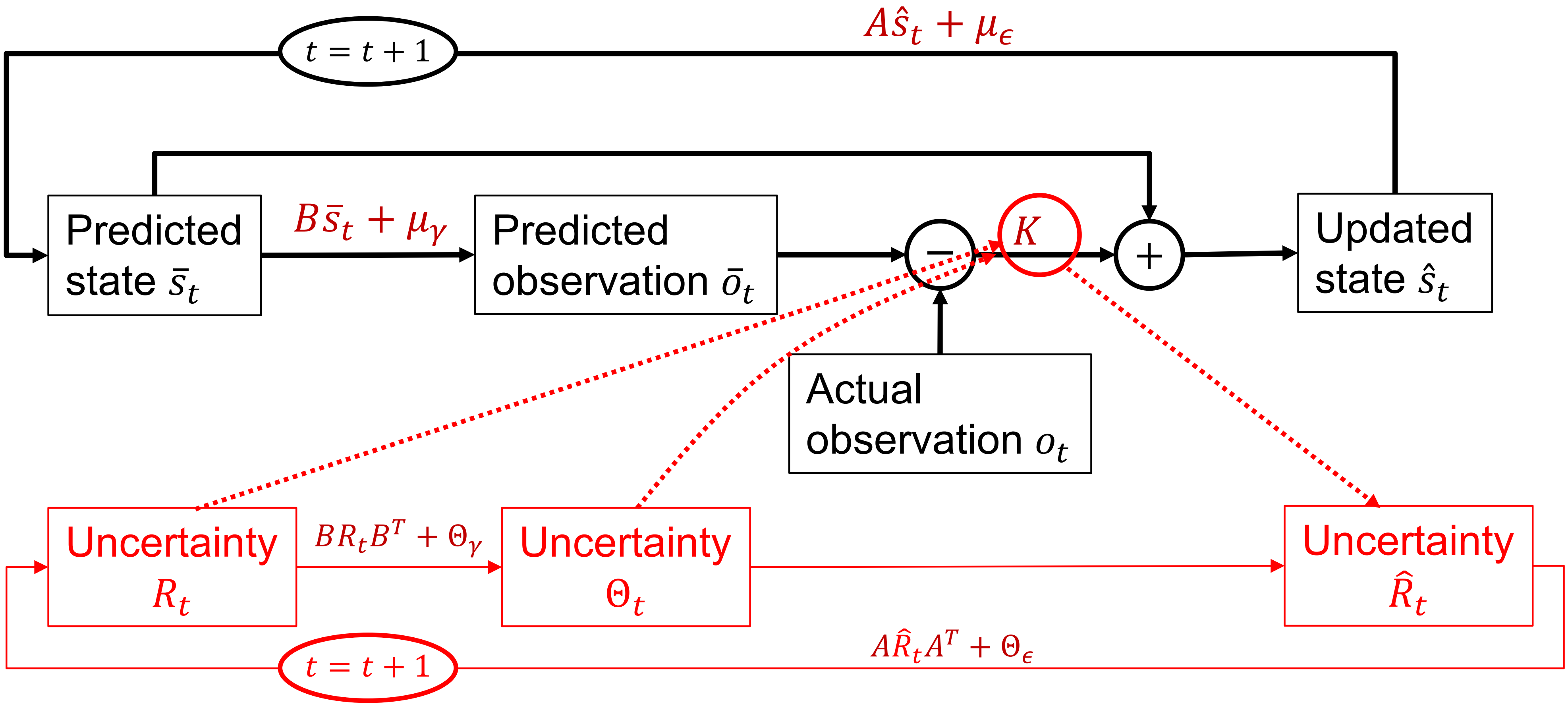
$$\hat{R}_t = \left(I - K_t B_t \right) R_t$$

Kalman filter



- Predict state
- Predict measurement
- Compute measurement error
- Update state

Kalman filter



- Predict state
- Predict measurement
- Compute measurement error
- Update state
- **Note: Progress of Kalman gain is not actually dependent on observations or estimated state…**

The Kalman Filter

- Very popular for tracking the state of processes
  - Control systems
  - Robotic tracking
    - Simultaneous localization and mapping
  - Radars
  - Even the stock market..
- What are the parameters of the process?

Kalman filter contd.

$$s_t = A_t s_{t-1} + \varepsilon_t$$

$$o_t = B_t s_t + \gamma_t$$

- Model parameters A and B must be known
  - Often the state equation includes an *additional* driving term: $s_t = A_t s_{t-1} + G_t u_t + \varepsilon_t$
  - The parameters of the driving term must be known
- The initial state distribution must be known

Defining the parameters

- State state must be carefully defined
  - E.g. for a robotic vehicle, the state is an extended vector that includes the current velocity and acceleration
    - $S = [X, dX, d^2X]$
- State equation: Must incorporate appropriate constraints
  - If state includes acceleration and velocity, velocity at next time = current velocity + acc. * time step
  - $St = AS_{t-1} + e$
    - $A = [1\ t\ 0.5t^2;\ \ 0\ 1\ t;\ 0\ 0\ 1]$

Parameters

- Observation equation:
  - Critical to have accurate observation equation
  - Must provide a valid relationship between state and observations
- Observations typically high-dimensional
  - May have higher or lower dimensionality than state

Problems

$$s_t = f(s_{t-1}, \varepsilon_t)$$

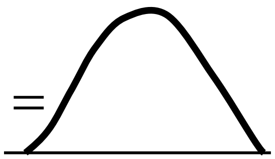
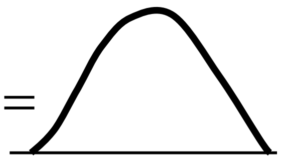
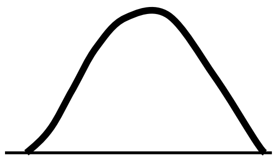
$$o_t = g(s_t, \gamma_t)$$

- f() and/or g() may not be nice linear functions
  - Conventional Kalman update rules are no longer valid
- ε and/or γ may not be Gaussian
  - Gaussian based update rules no longer valid

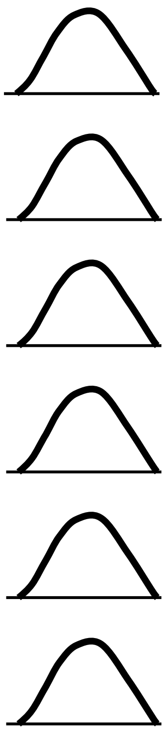
Linear Gaussian Model

$$s_t = A_t s_{t-1} + \varepsilon_t$$

$$o_t = B_t s_t + \gamma_t$$

$P(s) =$  $P(s_t|s_{t-1}) =$  $P(O_t|s_t) =$ 

a priori — Transition prob. — State output prob



$P(s_0) = P(s)$

$P(s_0| O_0) = C\, P(s_0)\, P(O_0| s_0)$

$$P(s_1 | \mathrm{O}_0) = \int_{-\infty}^{\infty} P(s_0 | \mathrm{O}_0) P(s_1 | s_0) ds_0$$

$P(s_1| O_{0:1}) = C\, P(s_1| O_0)\, P(O_1| s_0)$

$$P(s_2 | \mathrm{O}_{0:1}) = \int_{-\infty}^{\infty} P(s_1 | \mathrm{O}_{0:1}) P(s_2 | s_1) ds_1$$

$P(s_2| O_{0:2}) = C\, P(s_2| O_{0:1})\, P(O_2| s_2)$

All distributions remain Gaussian

Problems

$$s_t = f(s_{t-1}, \varepsilon_t)$$

$$o_t = g(s_t, \gamma_t)$$

- Nonlinear f() and/or g() : The Gaussian assumption breaks down
  - Conventional Kalman update rules are no longer valid

The problem with non-linear functions

$$s_t = f(s_{t-1}, \varepsilon_t)$$

$$o_t = g(s_t, \gamma_t)$$

$$P(s_t \mid \mathrm{o}_{0:t\text{-}1}) = \int_{-\infty}^{\infty} P(s_{t-1} \mid \mathrm{o}_{0:t\text{-}1}) P(s_t \mid s_{t-1}) ds_{t-1}$$

$$P(s_t \mid \mathrm{o}_{0:t}) = CP(s_t \mid \mathrm{o}_{0:t\text{-}1}) P(\mathrm{o}_t \mid s_t)$$

- Estimation requires knowledge of $P(o|s)$
  - Difficult to estimate for nonlinear $g()$
  - Even if it can be estimated, may not be tractable with update loop
- Estimation also requires knowledge of $P(s_t|s_{t-1})$
  - Difficult for nonlinear $f()$
  - May not be amenable to closed form integration

The problem with nonlinearity

$$o_t = g(s_t, \gamma_t)$$

- The PDF may not have a closed form

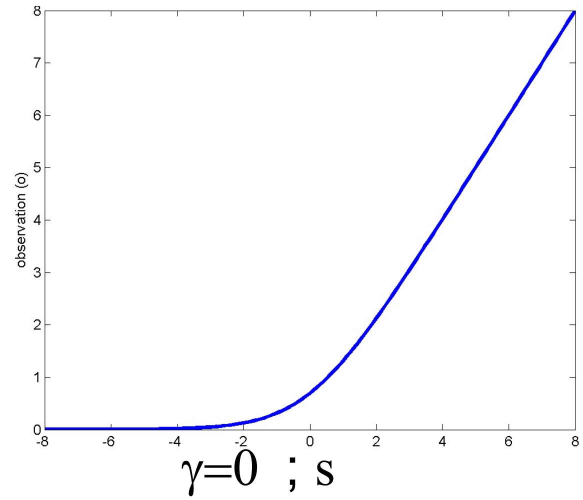
$$P(o_t \mid s_t) = \sum_{\gamma : g(s_t, \gamma) = o_t} \frac{P_\gamma(\gamma)}{| J_{g(s_t, \gamma)}(o_t) |}$$

$$| J_{g(s_t,\gamma)}(o_t) | = \begin{vmatrix} \frac{\partial o_t(1)}{\partial \gamma(1)} & \cdots & \frac{\partial o_t(1)}{\partial \gamma(n)} \\ \vdots & \ddots & \vdots \\ \frac{\partial o_t(n)}{\partial \gamma(1)} & \cdots & \frac{\partial o_t(n)}{\partial \gamma(n)} \end{vmatrix}$$

- Even if a closed form exists initially, it will typically become intractable very quickly

Example: a simple nonlinearity

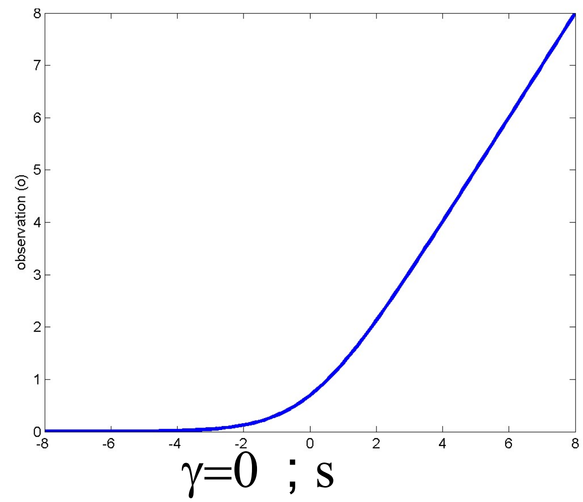
$$o = \gamma + \log(1 + \exp(s))$$



- $P(o|s) = ?$
  - Assume γ is Gaussian
  - $P(\gamma) = Gaussian(\gamma; \mu_\gamma, \Theta_\gamma)$

Example: a simple nonlinearity

$$o = \gamma + \log(1 + \exp(s))$$



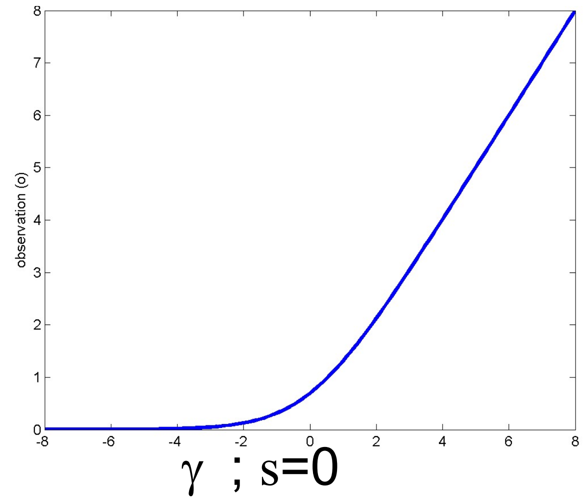
- P(o|s) = ?

$$P(\gamma) = Gaussian(\gamma; \mu_\gamma, \Theta_\gamma)$$

$$P(o \mid s) = Gaussian(o; \mu_\gamma + \log(1 + \exp(s)), \Theta_\gamma)$$

Example: At T=0.

$$o = \gamma + \log(1 + \exp(s))$$



- Assume initial probability P(s) is Gaussian

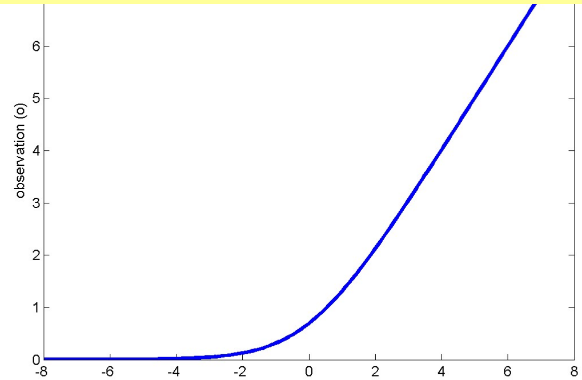
$$P(s_0) = P_0(s) = Gaussian(s; \bar{s}, R)$$

- Update $$P(s_0 \mid o_0) = CP(o_0 \mid s_0)P(s_0)$$

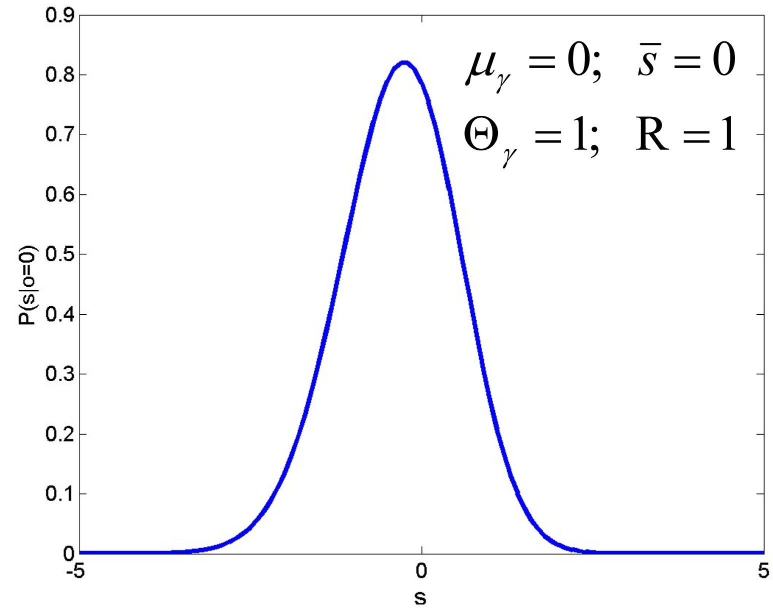
$$P(s_0 \mid o_0) = CGaussian(o; \mu_\gamma + \log(1 + \exp(s_0)), \Theta_\gamma)Gaussian(s_0; \bar{s}, R)$$

UPDATE: At T=0.

$$o = \gamma + \log(1 + \exp(s))$$



γ ; s=0



$$P(s_0 \mid o_0) = CGaussian(o; \mu_\gamma + \log(1 + \exp(s_0)), \Theta_\gamma) Gaussian(s_0; \bar{s}, R)$$

$$P(s_0 \mid o_0) = C \exp\left(\begin{matrix} -0.5(\mu_\gamma + \log(1 + \exp(s_0)) - o)^T \Theta_\gamma^{-1} (\mu_\gamma + \log(1 + \exp(s_0)) - o) \\ -0.5(s_0 - \bar{s})^T R^{-1} (s_0 - \bar{s}) \end{matrix} \right)$$

- = Not Gaussian

Prediction for T = 1

$$s_t = s_{t-1} + \varepsilon$$

$$P(\varepsilon) = Gaussian(\varepsilon; 0, \Theta_\varepsilon)$$

- Trivial, linear state transition equation

$$P(s_t \mid s_{t-1}) = Gaussian(s_t; s_{t-1}, \Theta_\varepsilon)$$

- Prediction $P(s_1 \mid o_0) = \int_{-\infty}^{\infty} P(s_0 \mid o_0) P(s_1 \mid s_0) ds_0$

$$P(s_1 \mid o_0) = \int_{-\infty}^{\infty} C \exp\begin{pmatrix} -0.5(\mu_\gamma + \log(1+\exp(s_0)) - o)^T \Theta_\gamma^{-1} (\mu_\gamma + \log(1+\exp(s_0)) - o) \\ -0.5(s_0 - \bar{s})^T R^{-1} (s_0 - \bar{s}) \end{pmatrix} \exp\left((s_1 - s_0)^T \Theta_\varepsilon^{-1} (s_1 - s_0)\right) ds_0$$

- = intractable

Update at T=1 and later

- Update at T=1

$$P(s_t \mid \mathrm{o}_{0:t}) = CP(s_t \mid \mathrm{o}_{0:t-1})P(\mathrm{o}_t \mid s_t)$$

  – Intractable

- Prediction for T=2

$$P(s_t \mid \mathrm{o}_{0:t-1}) = \int_{-\infty}^{\infty} P(s_{t-1} \mid \mathrm{o}_{0:t-1})P(s_t \mid s_{t-1})ds_{t-1}$$

  – Intractable

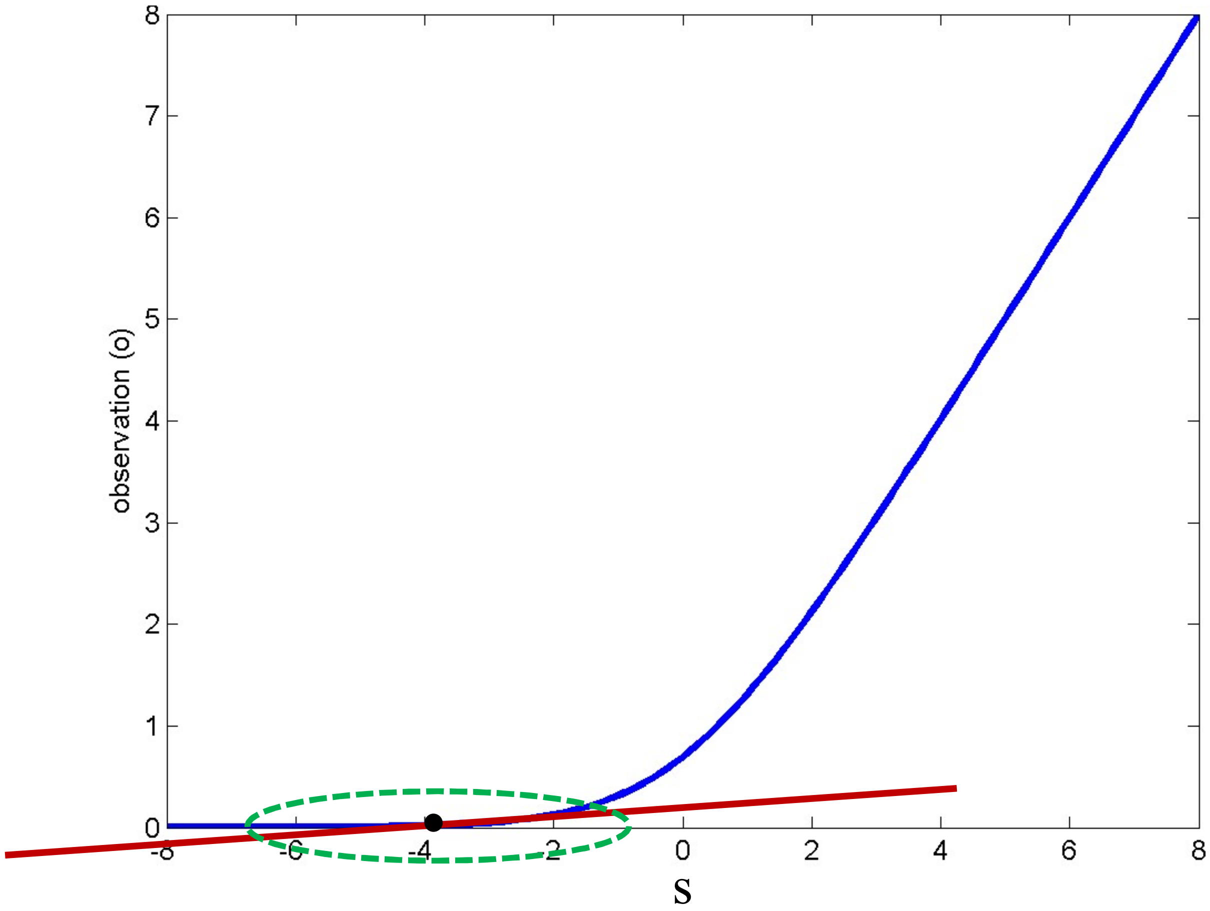
The State prediction Equation

$$s_t = f(s_{t-1}, \varepsilon_t)$$

- Similar problems arise for the state prediction equation

- $P(s_t|s_{t-1})$ may not have a closed form
- Even if it does, it may become intractable within the prediction and update equations
  - Particularly the prediction equation, which includes an integration operation

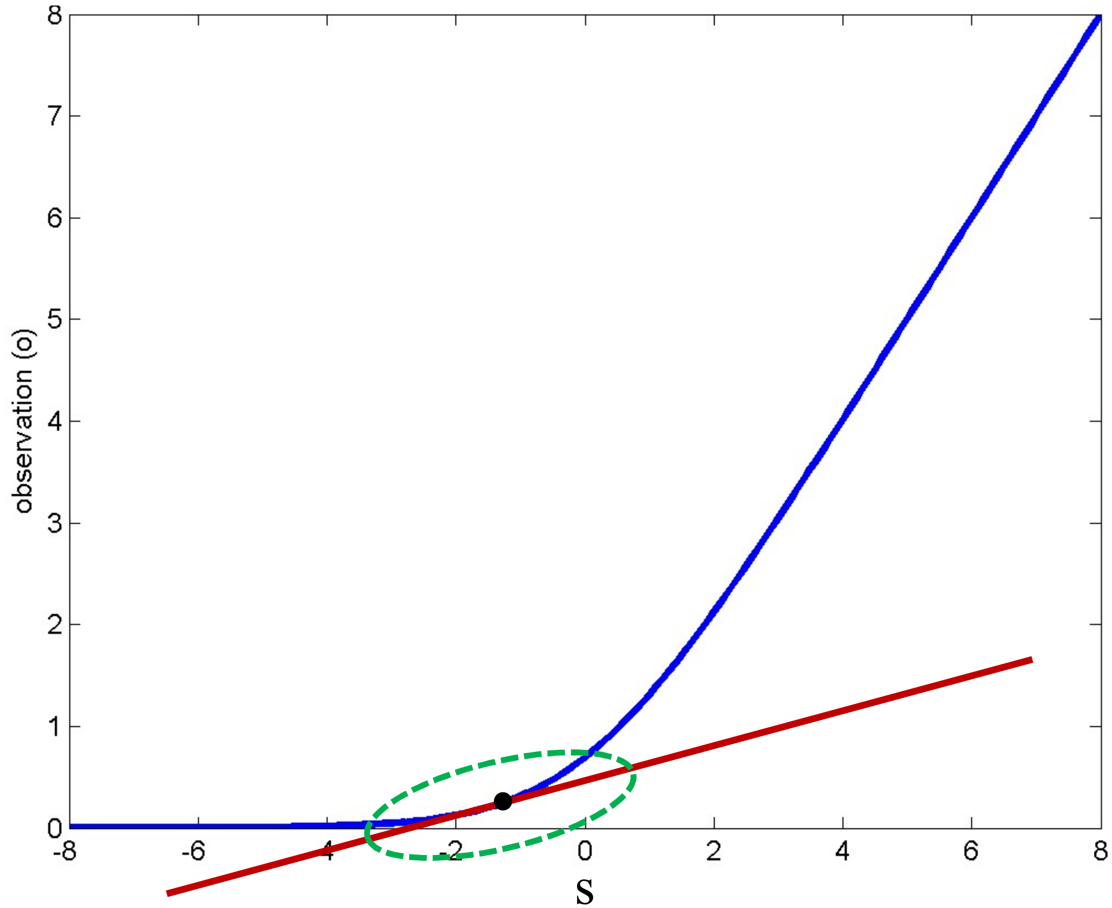
Simplifying the problem: Linearize



$$o = \gamma + \log(1 + \exp(s))$$

- The *tangent* at any point is a good *local* approximation if the function is sufficiently smooth

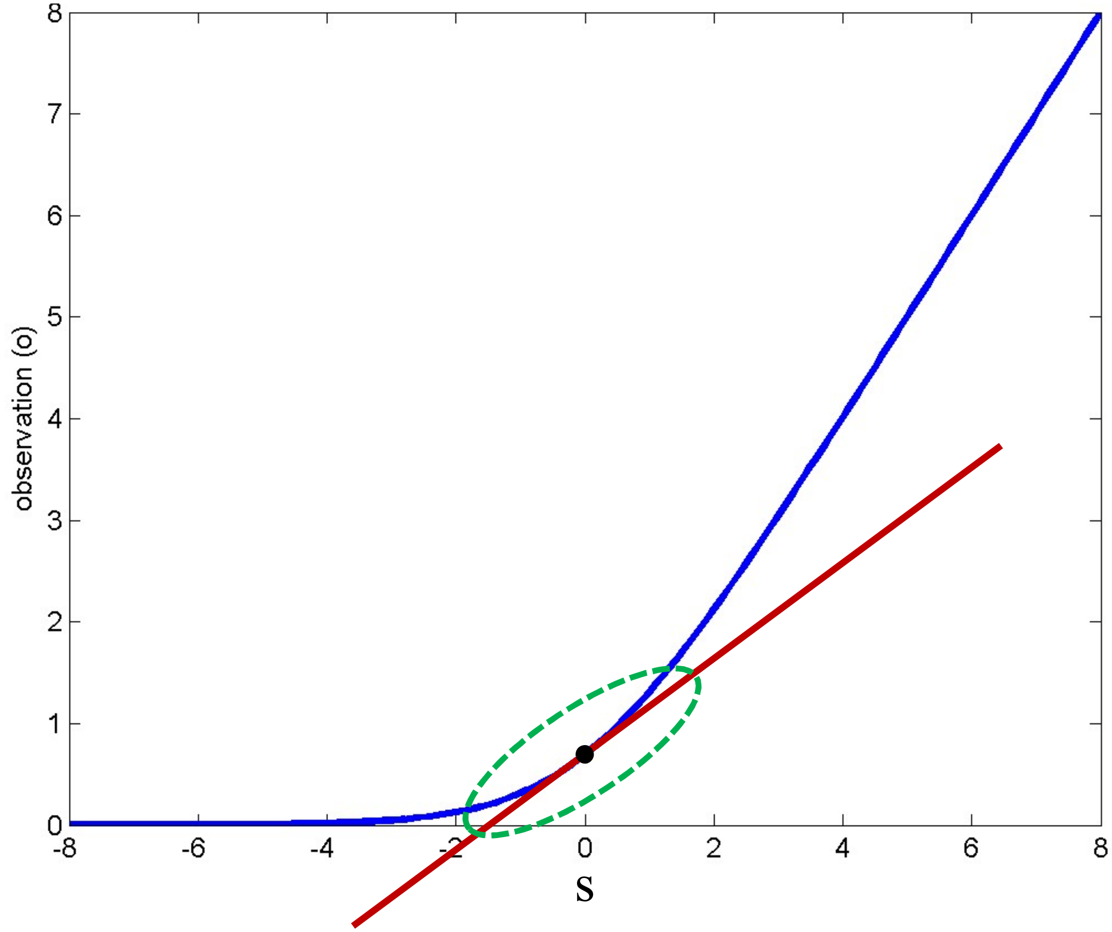
Simplifying the problem: Linearize



$$o = \gamma + \log(1 + \exp(s))$$

- The *tangent* at any point is a good *local* approximation if the function is sufficiently smooth

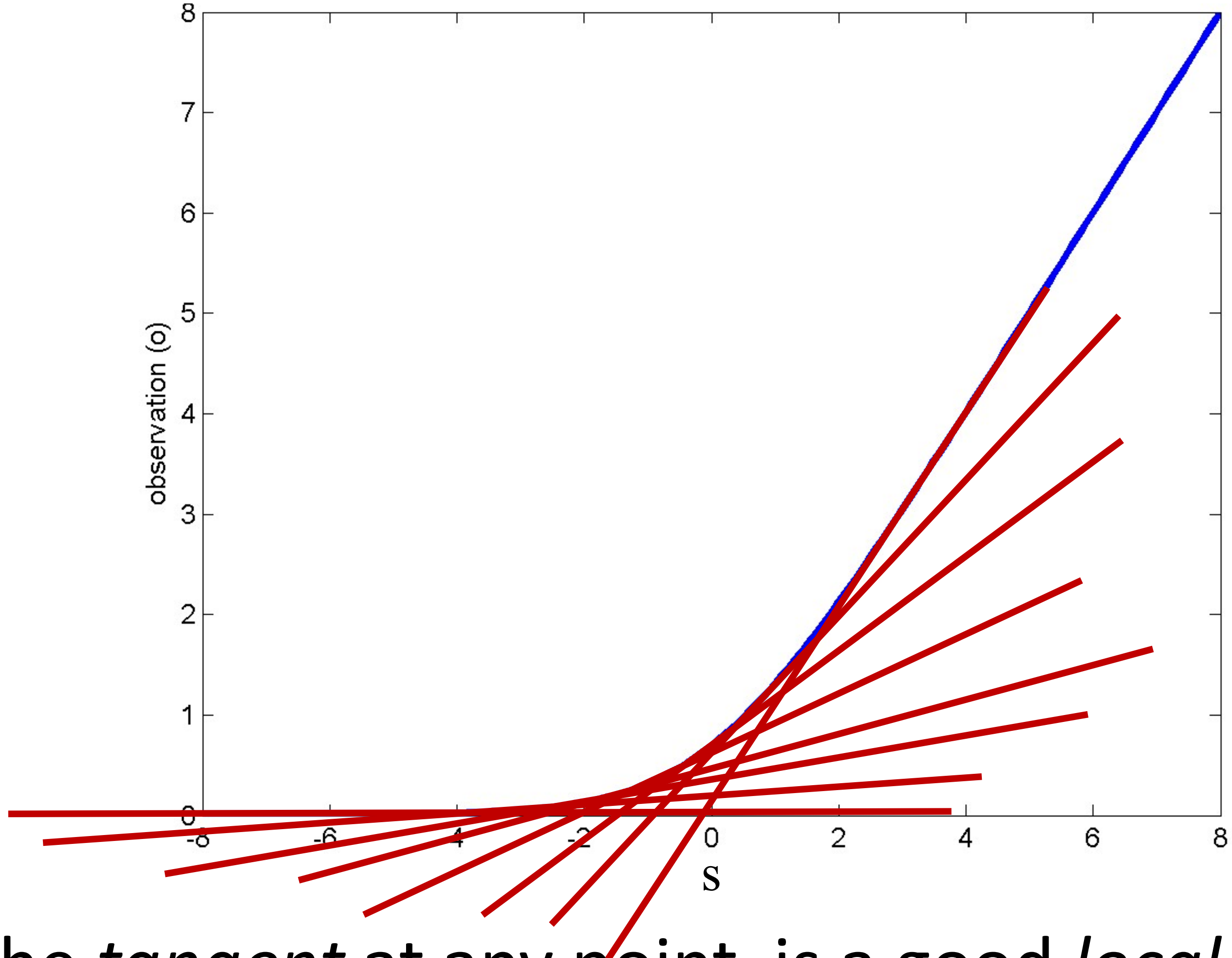
Simplifying the problem: Linearize



$$o = \gamma + \log(1 + \exp(s))$$

- The *tangent* at any point is a good *local* approximation if the function is sufficiently smooth

Simplifying the problem: Linearize



- The *tangent* at any point is a good *local* approximation if the function is sufficiently smooth

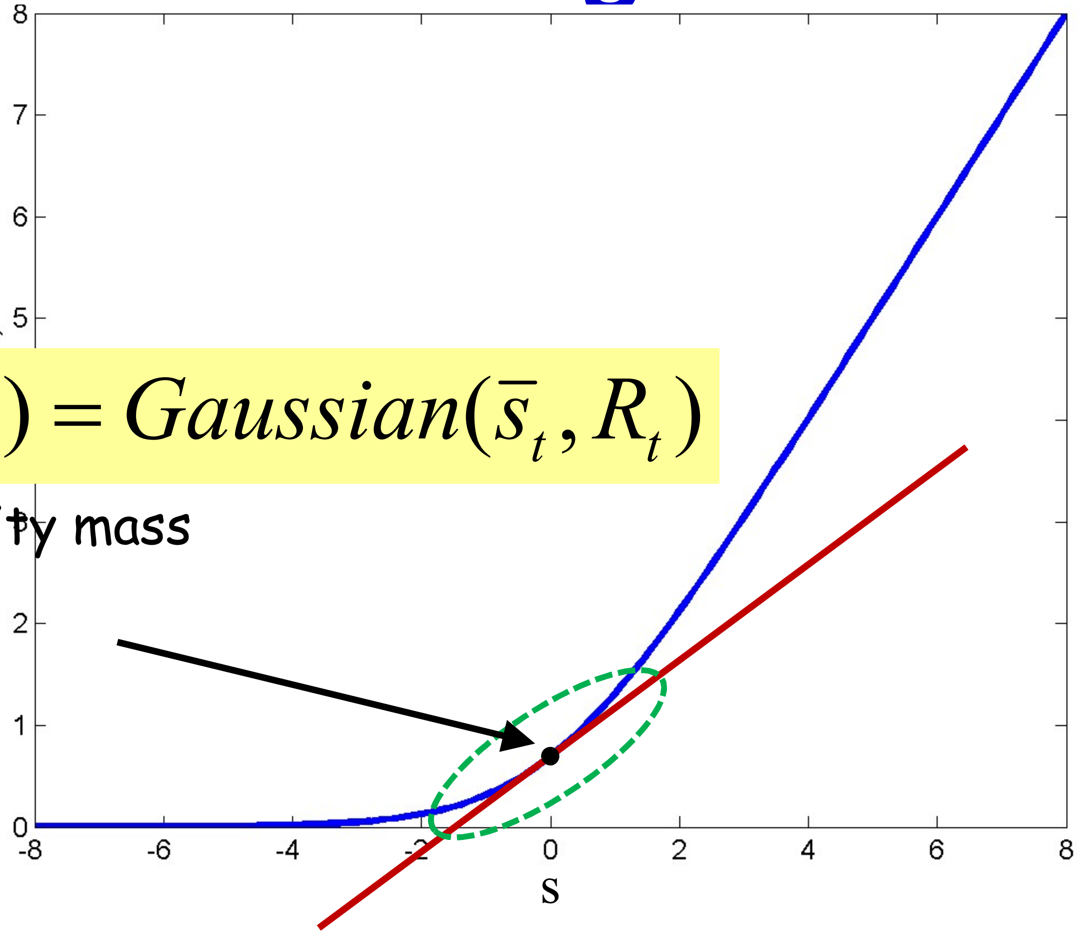
Linearizing the observation function

$$P(s_t \mid o_{0:t-1}) = Gaussian(\bar{s}_t, R_t)$$

$$o = \gamma + g(s) \quad \Rightarrow \quad o \approx \gamma + g(\bar{s}_t) + J_g(\bar{s}_t)(s - \bar{s}_t)$$

- Simple first-order Taylor series expansion
  - J() is the Jacobian matrix
    - Simply a determinant for scalar state
- Expansion around *current* predicted *a priori* (or predicted) mean of the state
  - Linear approximation changes with time

Most probability is in the low-error region



$$P(s_t \mid o_{0:t-1}) = Gaussian(\bar{s}_t, R_t)$$

Most probability mass close to mean

- $P(s_t)$ is small where approximation error is large
  - Most of the probability mass of *s* is in low-error regions

Linearizing the observation function

$$P(s_t \mid o_{0:t-1}) = Gaussian(\bar{s}_t, R_t)$$

$$o = \gamma + g(s) \quad \Rightarrow \quad o \approx \gamma + g(\bar{s}_t) + J_g(\bar{s}_t)(s - \bar{s}_t)$$

- With the linearized approximation the system becomes "linear"
- The observation PDF becomes Gaussian

$$P(\gamma) = Gaussian(\gamma; 0, \Theta_\gamma)$$

$$P(o \mid s) = Gaussian(o; g(\bar{s}) + J_g(\bar{s})(s - \bar{s}), \Theta_\gamma)$$

The state equation?

$$s_t = f(s_{t-1}) + \varepsilon \qquad P(\varepsilon) = Gaussian(\varepsilon; 0, \Theta_\varepsilon)$$

- Again, direct use of f() can be disastrous

- Solution: Linearize

$$P(s_{t-1} \mid o_{0:t-1}) = Gaussian(s_{t-1}; \hat{s}_{t-1}, \hat{R}_{t-1})$$

$$s_t = f(s_{t-1}) + \varepsilon \quad \Rightarrow \quad s_t \approx \varepsilon + f(\hat{s}_{t-1}) + J_f(\hat{s}_{t-1})(s_{t-1} - \hat{s}_{t-1})$$

- Linearize around the mean of the updated distribution of s at $t-1$
  - Converts the system to a linear one

Linearized System

$$o = \gamma + g(s)$$

$$s_t = f(s_{t-1}) + \varepsilon$$

$$o \approx \gamma + g(\bar{s}_t) + J_g(\bar{s}_t)(s - \bar{s}_t)$$

$$s_t \approx \varepsilon + f(\hat{s}_{t-1}) + J_f(\hat{s}_{t-1})(s_{t-1} - \hat{s}_{t-1})$$

- Now we have a simple time-varying linear system
- Kalman filter equations directly apply

The Extended Kalman filter

$$s_t = f(s_{t-1}) + \varepsilon$$

$$o_t = g(s_t) + \gamma$$

- Prediction

$$\bar{s}_t = f(\hat{s}_{t-1})$$

$$R_t = \Theta_\varepsilon + A_t \hat{R}_{t-1} A_t^T$$

$$A_t = J_f(\hat{s}_{t-1})$$

$$B_t = J_g(\bar{s}_t)$$

Jacobians used in Linearization

- Update

$$K_t = R_t B_t^T \left(B_t R_t B_t^T + \Theta_\gamma \right)^{-1}$$

$$\hat{s}_t = \bar{s}_t + K_t \left(o_t - g(\bar{s}_t) \right)$$

$$\hat{R}_t = \left(I - K_t B_t \right) R_t$$

Assuming ε and γ are 0 mean for simplicity

The Extended Kalman filter

- Prediction

$$s_t = f(s_{t-1}) + \varepsilon$$

$$\bar{s}_t = f(\hat{s}_{t-1})$$

$$o_t = g(s_t) + \gamma$$

The *predicted* state at time t is obtained simply by propagating the estimated state at t-1 through the state dynamics equation

$$K_t = R_t B_t^T (B_t R_t B_t^T + \Theta_\gamma)^{-1}$$

$$\hat{s}_t = \bar{s}_t + K_t (o_t - g(\bar{s}_t))$$

$$\hat{R}_t = (I - K_t B_t) R_t$$

The Extended Kalman filter

- Prediction

$$\bar{s}_t = f(\hat{s}_{t-1})$$

$$R_t = \Theta_\varepsilon + A_t \hat{R}_{t-1} A_t^T$$

$$s_t = f(s_{t-1}) + \varepsilon$$

$$o_t = g(s_t) + \varepsilon$$

$$A_t = J_f(\hat{s}_{t-1})$$

$$B_t = J_g(\bar{s}_t)$$

Uncertainty of prediction.
The variance of the predictor = variance of ε_t + variance of As_{t-1}

A is obtained by linearizing f()

$$R_t = (I - K_t B_t) R_t$$

The Extended Kalman filter

- Prediction

$$s_t = f(s_{t-1}) + \varepsilon$$

$$\bar{s}_t = f(\hat{s}_{t-1})$$

$$o_t = g(s_t) + \varepsilon$$

$$R_t = \Theta_\varepsilon + A_t \hat{R}_{t-1} A_t^T$$

- Update

$$B_t = J_g(\bar{s}_t)$$

$$K_t = R_t B_t^T \left(B_t R_t B_t^T + \Theta_\gamma\right)^{-1}$$

The Kalman gain is the slope of the MAP estimator that predicts s from o

$RB^T = C_{so}$, $(BRB^T + \Theta) = C_{oo}$

B is obtained by linearizing g()

The Extended Kalman filter

- Prediction

$$s_t = f(s_{t-1}) + \varepsilon$$

$$\bar{s}_t = f(\hat{s}_{t-1})$$

$$o_t = g(s_t) + \varepsilon$$

$$R_t = \Theta_\varepsilon + A_t \hat{R}_{t-1} A_t^T$$

We can also predict the *observation* from the predicted state using the observation equation

$$\hat{s}_t = \bar{s}_t + K_t\left(o_t - g(\bar{s}_t)\right)$$

$$\hat{R}_t = \left(I - K_t B_t\right) R_t$$

$$\bar{o}_t = g(\bar{s}_t)$$

The Extended Kalman filter

$$s_t = f(s_{t-1}) + \varepsilon$$

$$o_t = g(s_t) + \varepsilon$$

- Prediction

$$\bar{s}_t = f(\hat{s}_{t-1})$$

$$R_t = \Theta_\varepsilon + A_t \hat{R}_{t-1} A_t^T$$

We must correct the predicted value of the state after making an observation

$$\bar{o}_t = g(\bar{s}_t)$$

$$\hat{s}_t = \bar{s}_t + K_t\left(o_t - g(\bar{s}_t)\right)$$

The correction is the difference between the *actual* observation and the *predicted* observation, scaled by the Kalman Gain

The Extended Kalman filter

- Prediction

$$s_t = f(s_{t-1}) + \varepsilon$$

$$o_t = g(s_t) + \varepsilon$$

$$\bar{s}_t = f(\hat{s}_{t-1})$$

$$R_t = \Theta_\varepsilon + A_t \hat{R}_{t-1} A_t^T$$

$$B_t = J_g(\bar{s}_t)$$

The uncertainty in state decreases if we observe the data and make a correction

The reduction is a multiplicative "shrinkage" based on Kalman gain and B

$$\hat{R}_t = (I - K_t B_t) R_t$$

The Extended Kalman filter

$$s_t = f(s_{t-1}) + \varepsilon$$

$$o_t = g(s_t) + \varepsilon$$

$$A_t = J_f(\hat{s}_{t-1})$$

$$B_t = J_g(\bar{s}_t)$$

- Prediction

$$\bar{s}_t = f(\hat{s}_{t-1})$$

$$R_t = \Theta_\varepsilon + A_t \hat{R}_{t-1} A_t^T$$

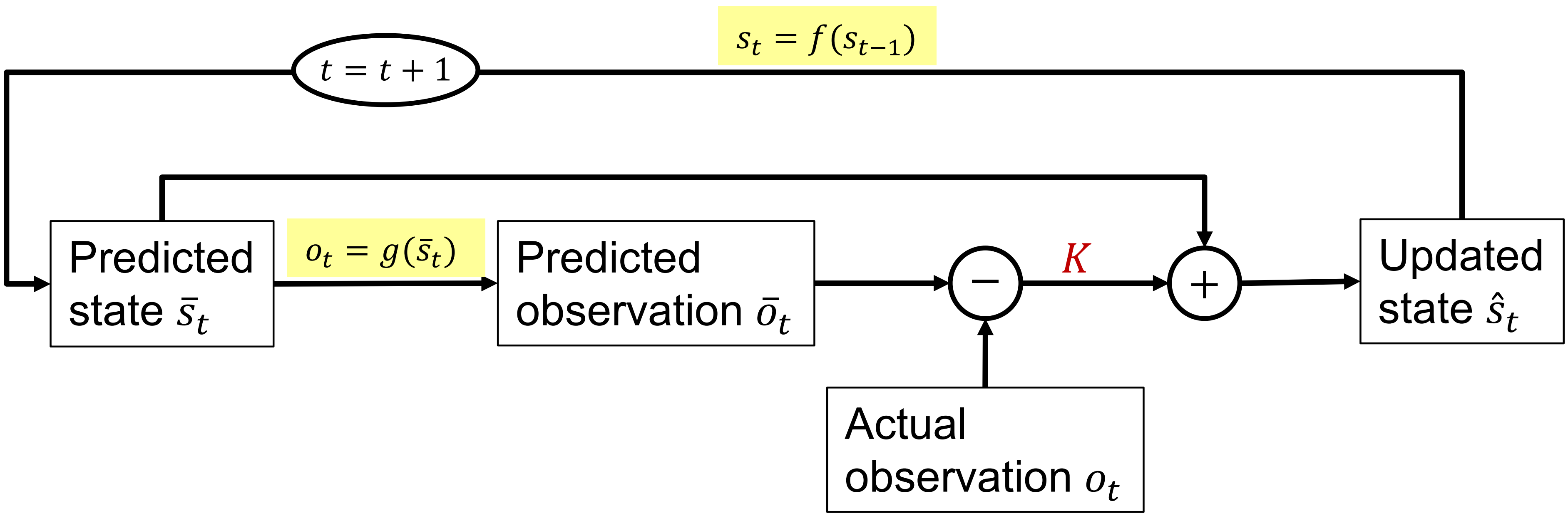
- Update

$$K_t = R_t B_t^T \left(B_t R_t B_t^T + \Theta_\gamma \right)^{-1}$$

$$\hat{s}_t = \bar{s}_t + K_t \left(o_t - g(\bar{s}_t) \right)$$

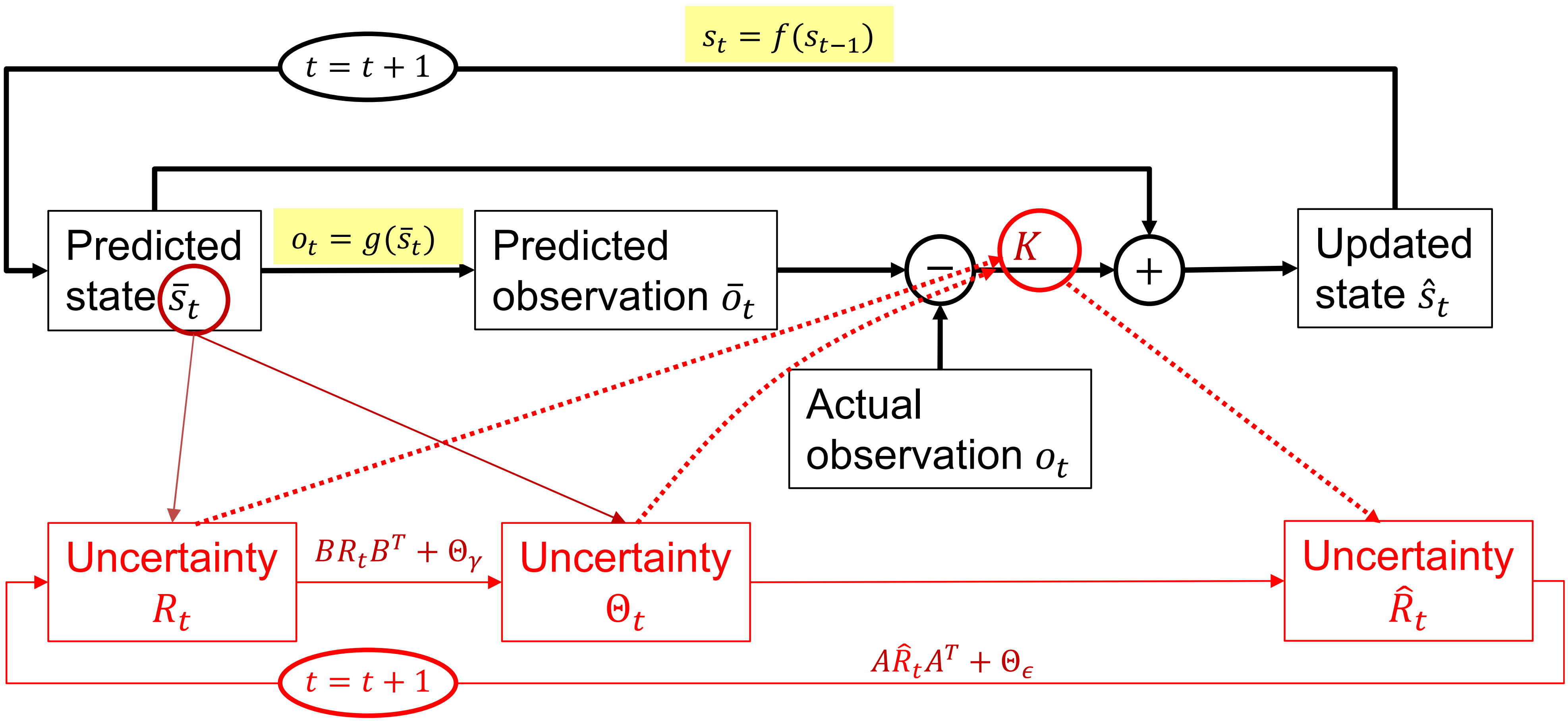
$$\hat{R}_t = \left(I - K_t B_t \right) R_t$$

Extended Kalman filter



- Predict state
- Predict measurement
- Compute measurement error
- Update state

Kalman filter

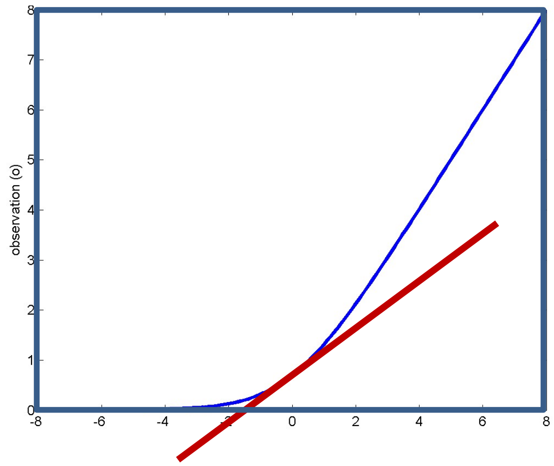


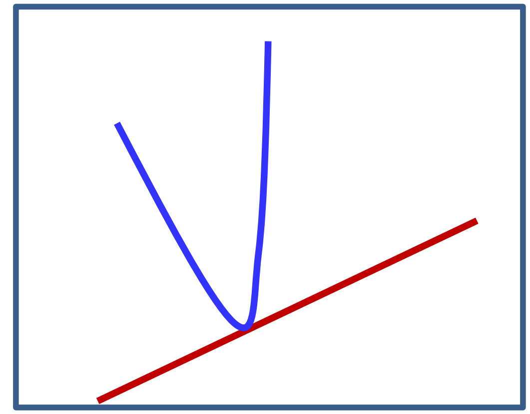
- Predict state
- Predict measurement
- Compute measurement error
- Update state
- **Note: Progress of Kalman gain is dependent on estimated state through the Jacobian…**

EKFs

- EKFs are probably the most commonly used algorithm for tracking and prediction
  - Most systems are non-linear
  - Specifically, the relationship between state and observation is usually nonlinear
  - The approach can be extended to include non-linear functions of noise as well
- The term "Kalman filter" often simply refers to an *extended* Kalman filter in most contexts.
- But..

EKFs have limitations





- If the non-linearity changes too quickly with s, the linear approximation is invalid
  - Unstable
- The estimate is often biased
  - The true function lies entirely on one side of the approximation
- Various extensions have been proposed:
  - Invariant extended Kalman filters (IEKF)
  - Unscented Kalman filters (UKF)

Conclusions

- HMMs are predictive models
- Continuous-state models are simple extensions of HMMs
  - Same math applies
- Prediction of linear, Gaussian systems can be performed by Kalman filtering
- Prediction of non-linear, Gaussian systems can be performed by Extended Kalman filtering