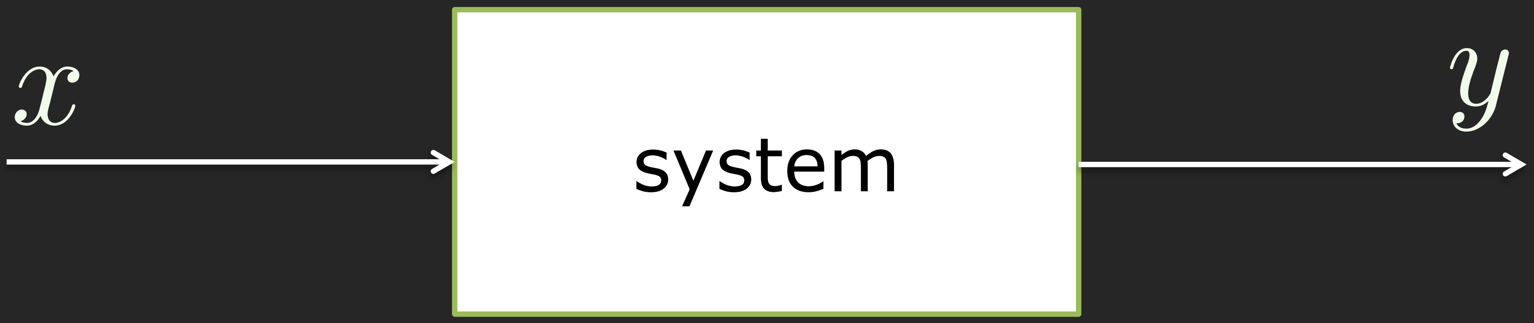
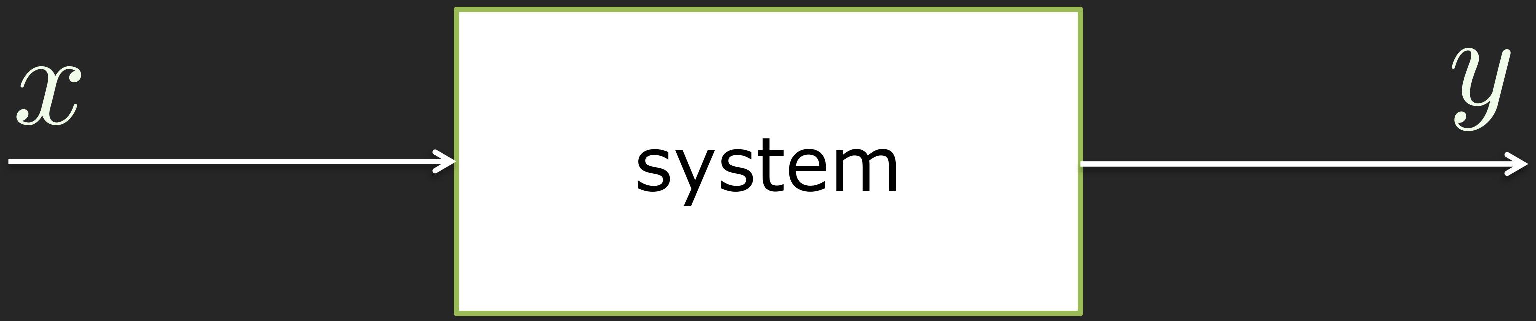
Introduction to Compressive Sensing

Aswin Sankaranarayanan



$$y = \Phi x$$

Is this system linear ?



x

system

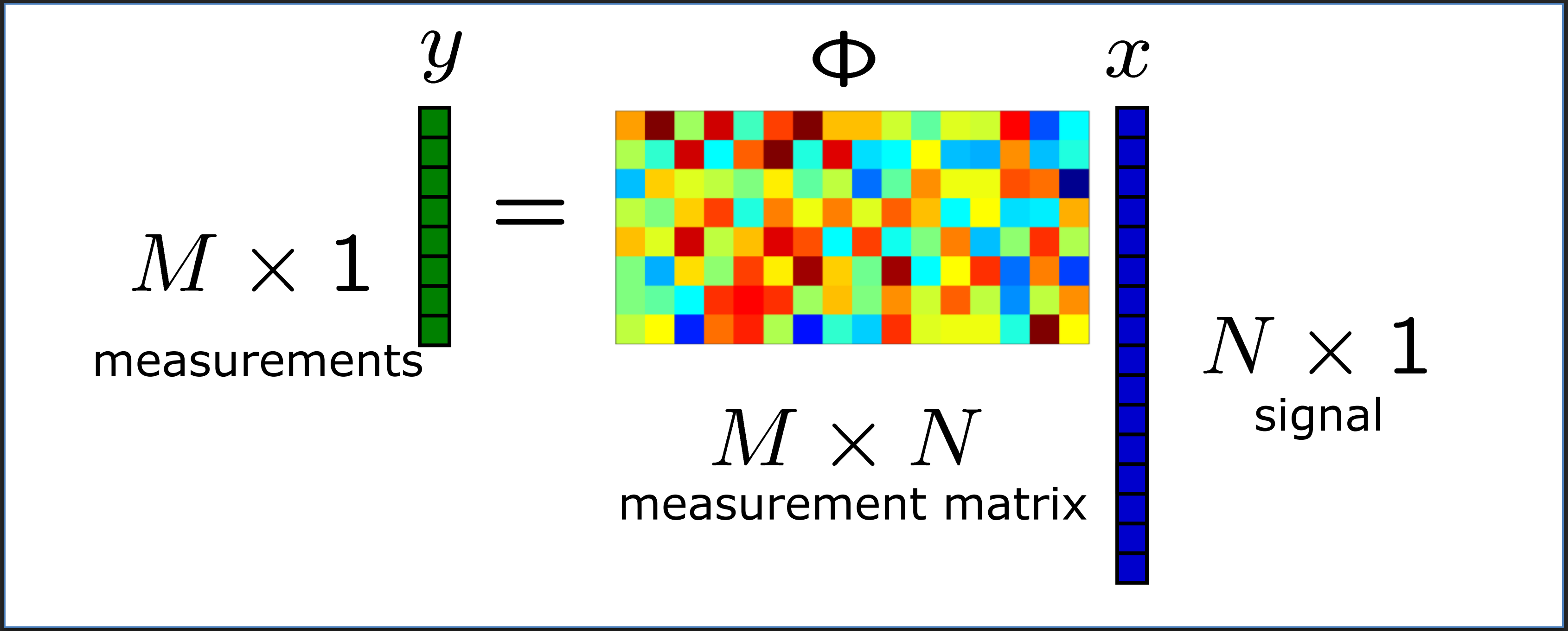
y

$$y = \Phi x$$

Is this system linear ?

Given *y*, can we recovery *x* ?

Under-determined problems



y = Φ x

$M \times 1$

measurements

$M \times N$

measurement matrix

$N \times 1$

signal

If M < N, then the system is information lossy

Image credit
Graeme Pope

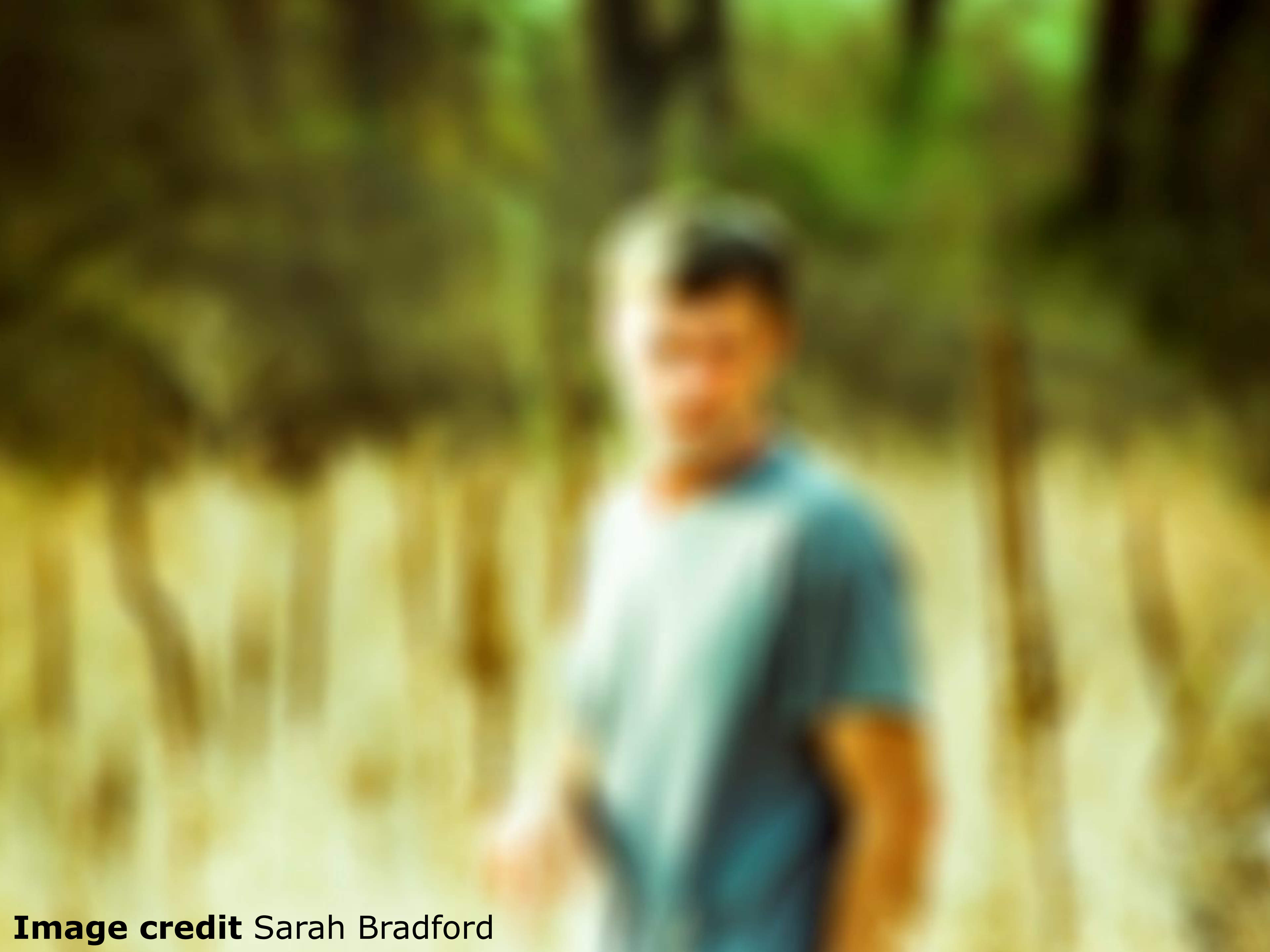
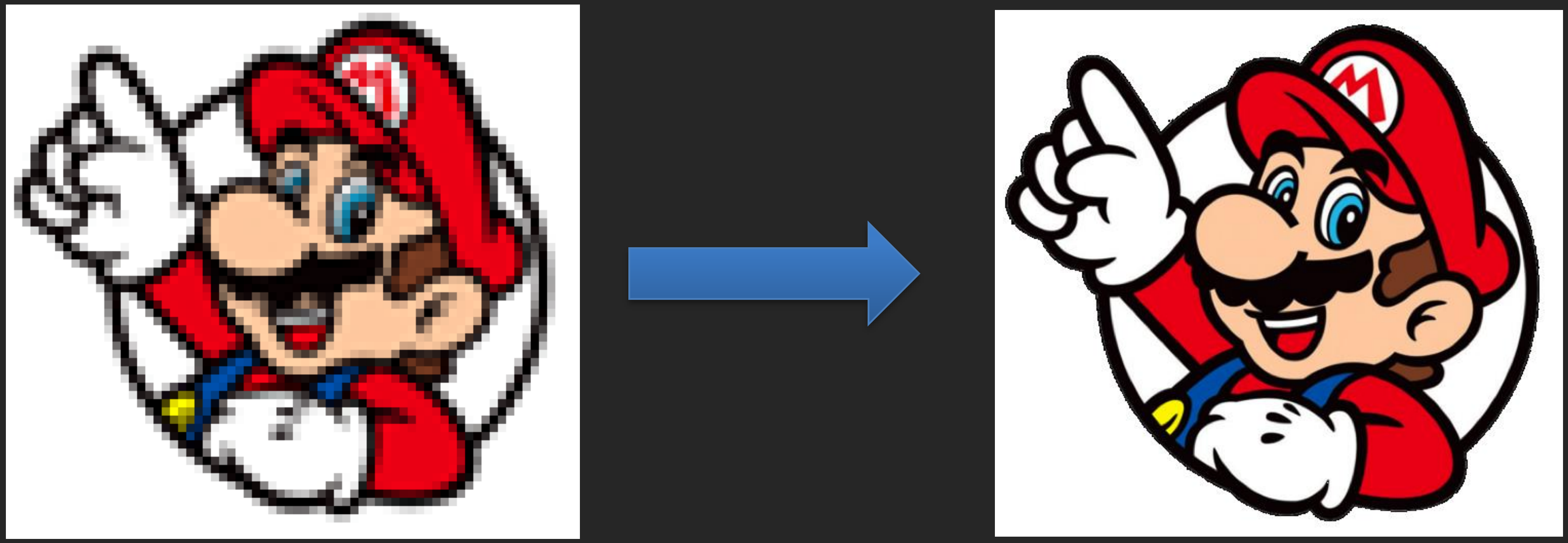


Image credit Sarah Bradford

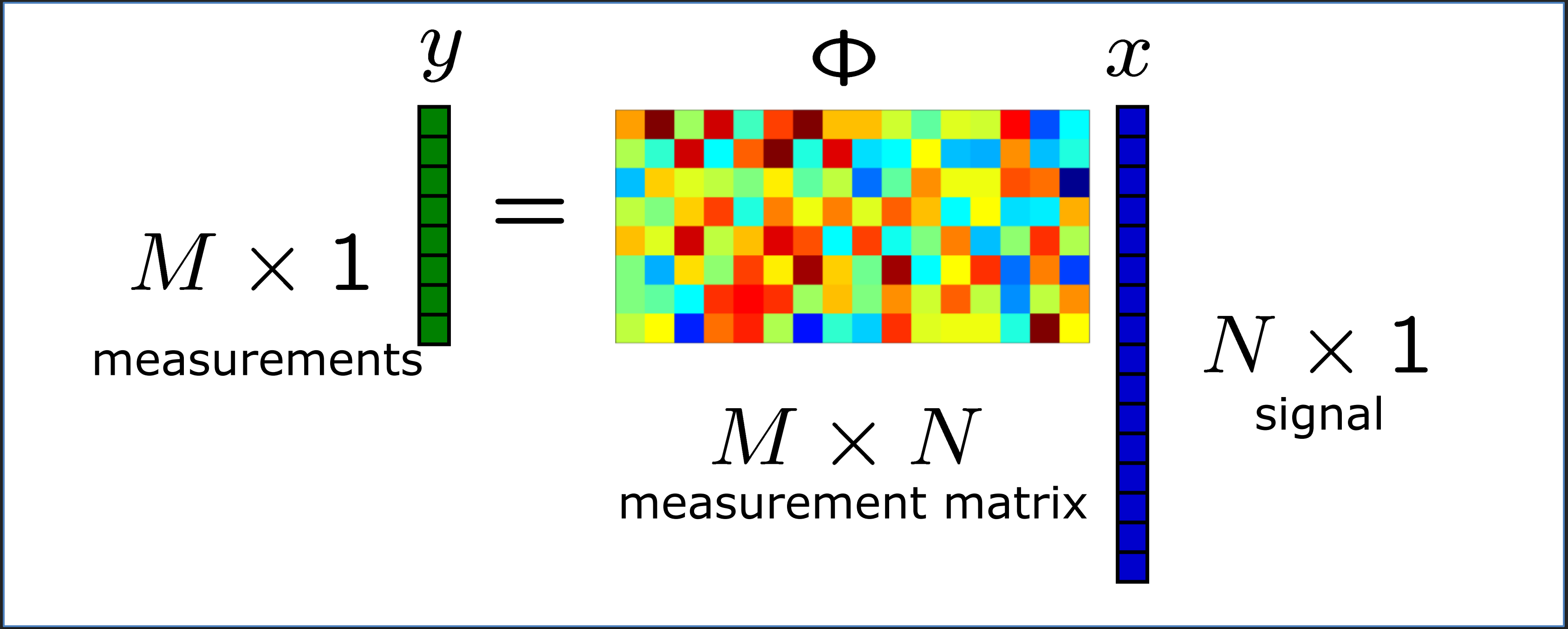
Super-resolution



Can we increase the resolution of this image ?

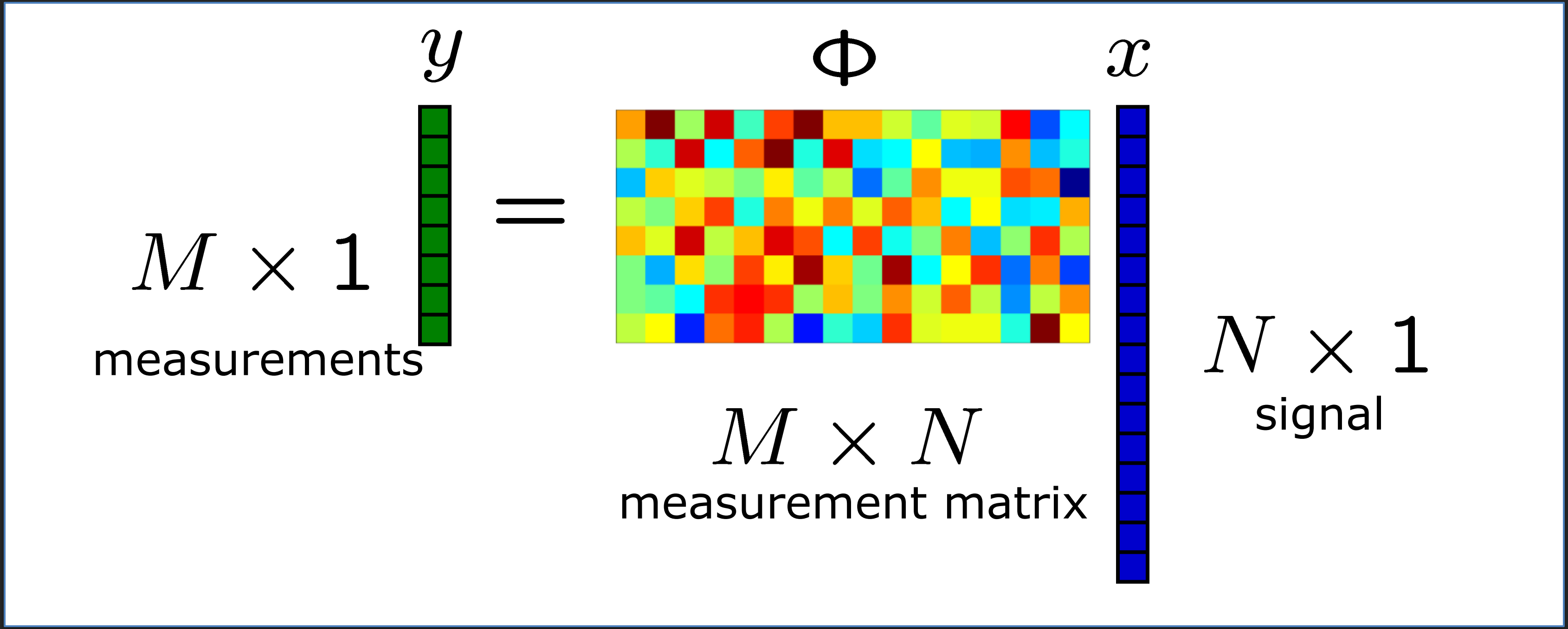
[(Link: Depixelizing pixel art)]

Under-determined problems



Fewer knowns than unknowns!

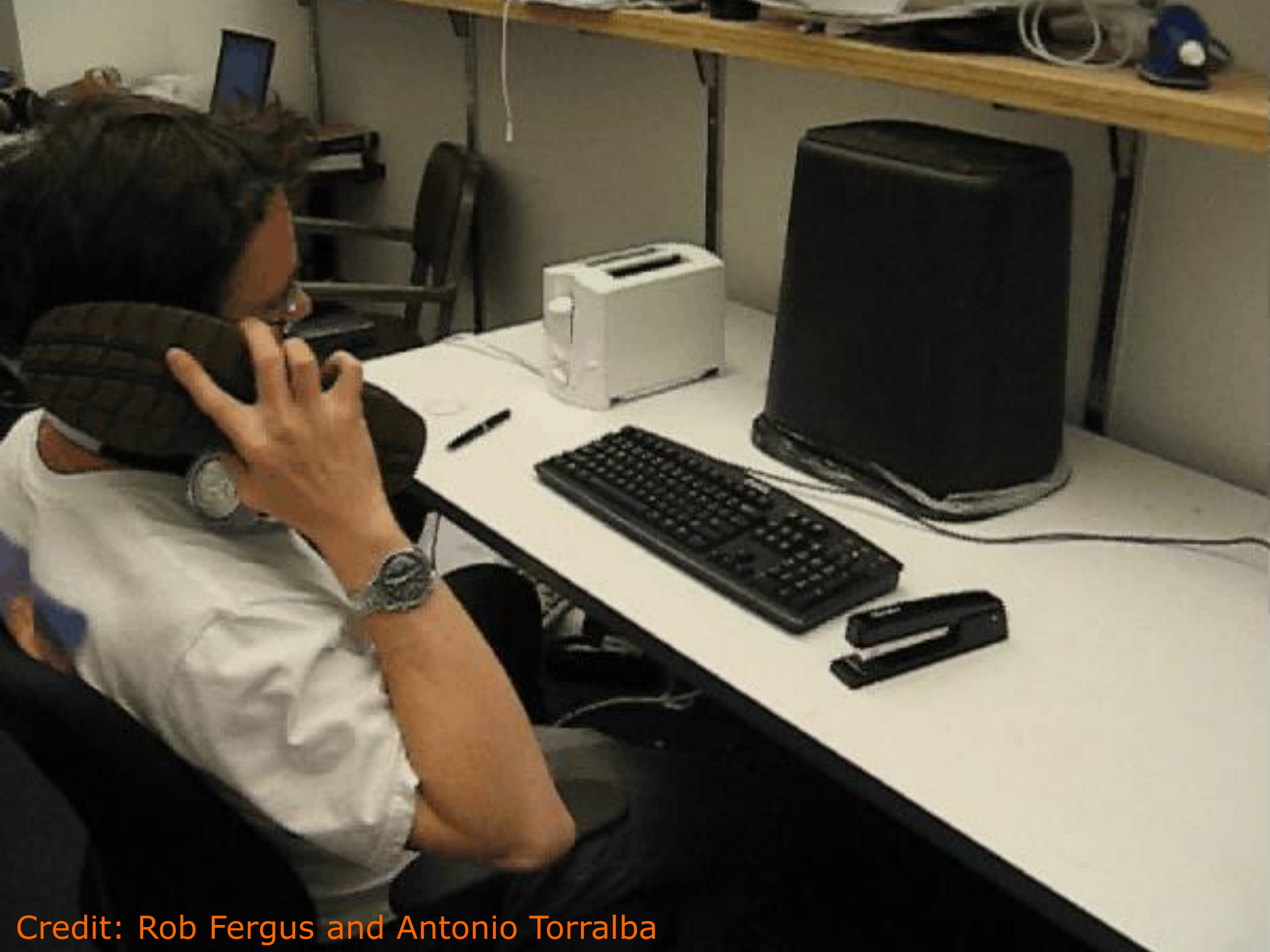
Under-determined problems



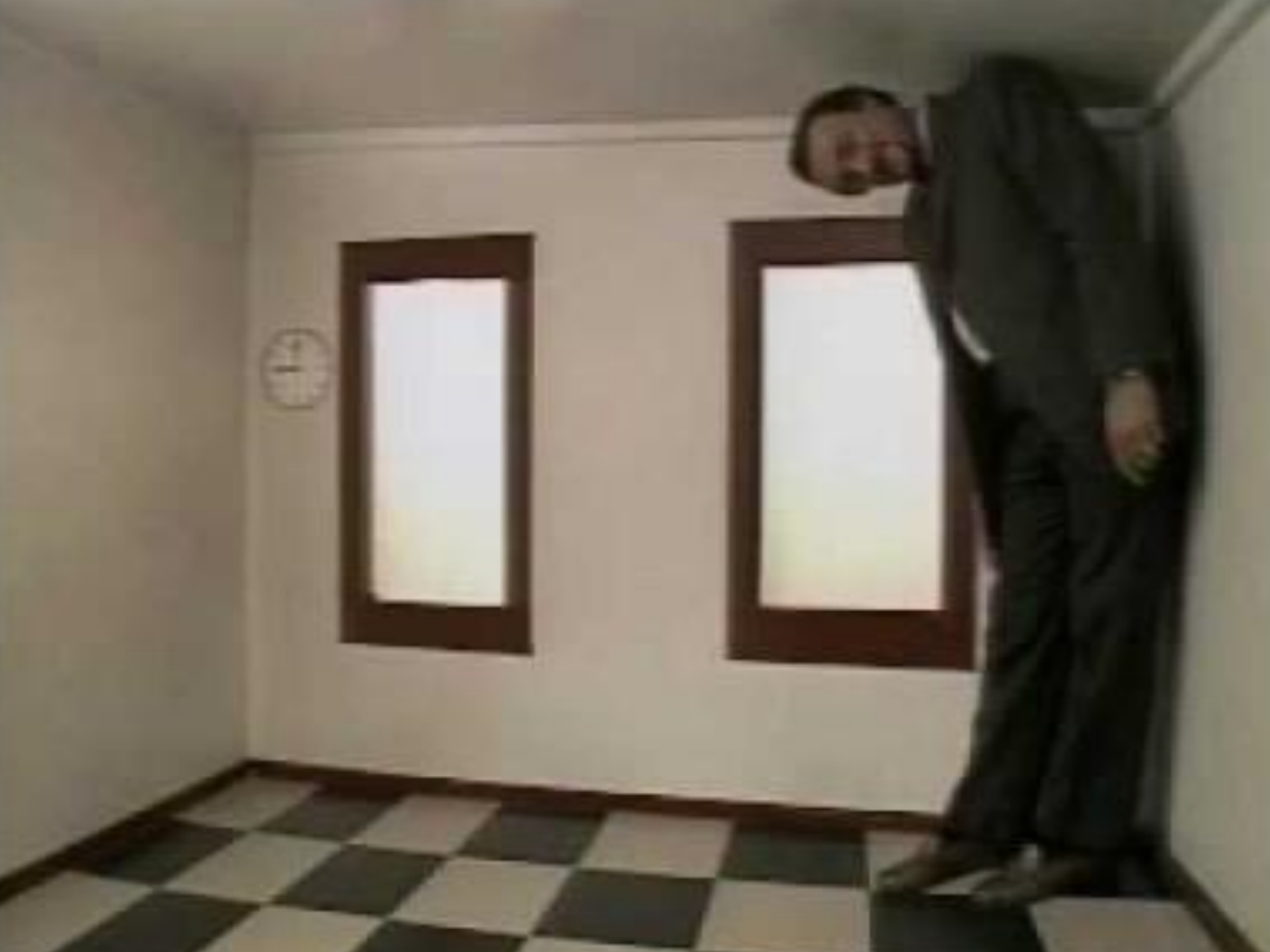
Fewer knowns than unknowns!

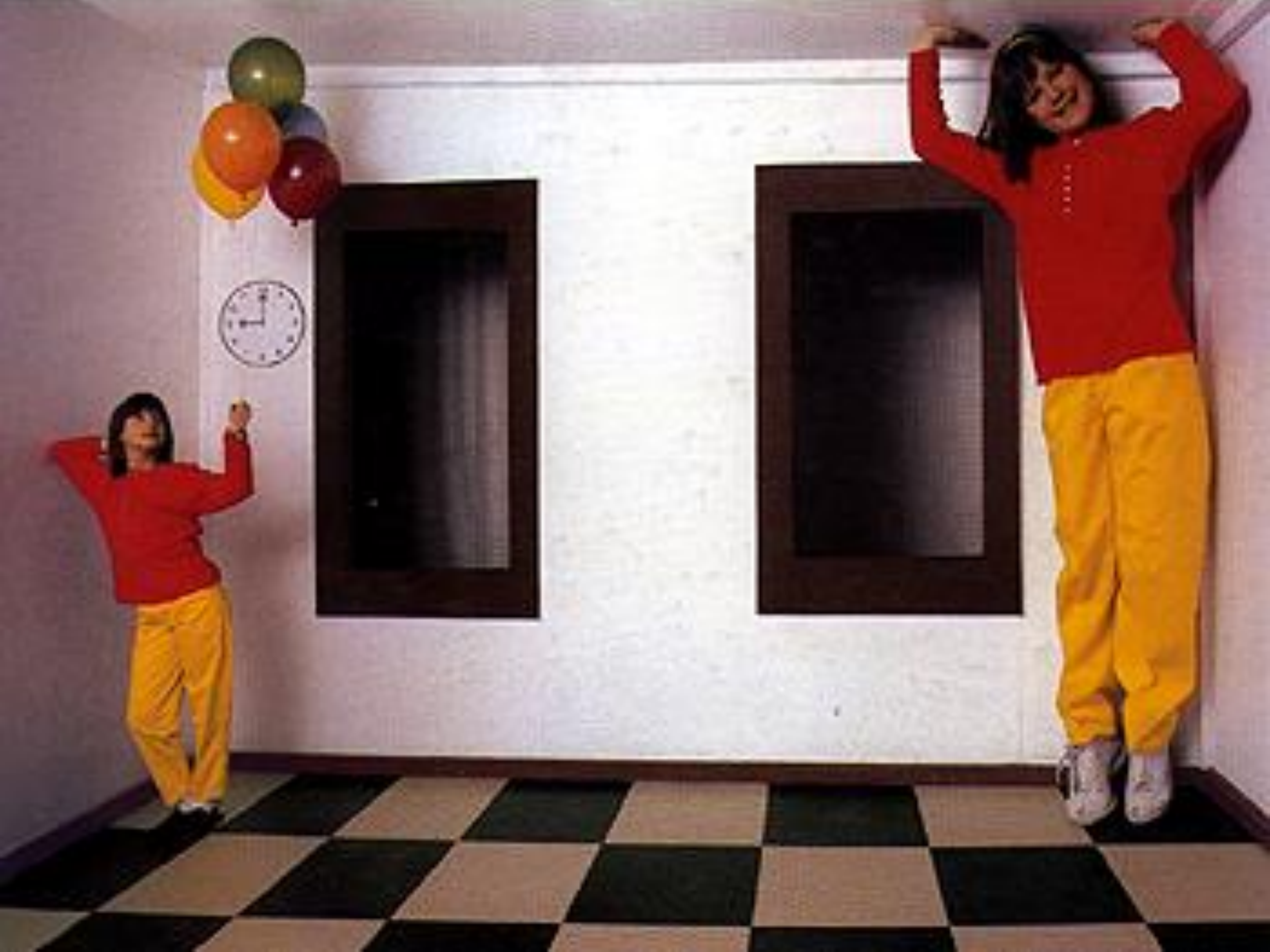
An infinite number of solutions to such problems

Credit: Rob Fergus and Antonio Torralba



Credit: Rob Fergus and Antonio Torralba





Is there anything we can do about this ?

Complete the sentences

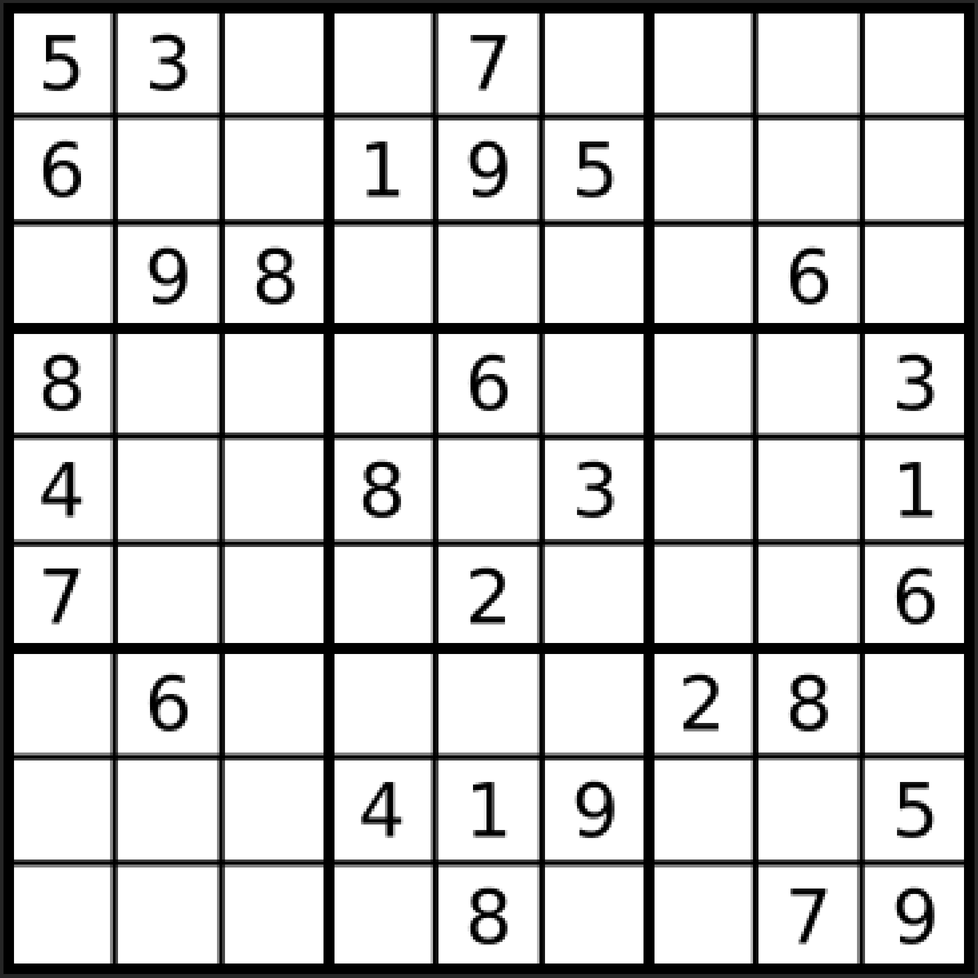
Bksh – th mn, th myth, th lgnd

Gv m frdm, r gv m dth

Hy, I m slvng n ndr-dtrmnd lnr systm.

how: ?

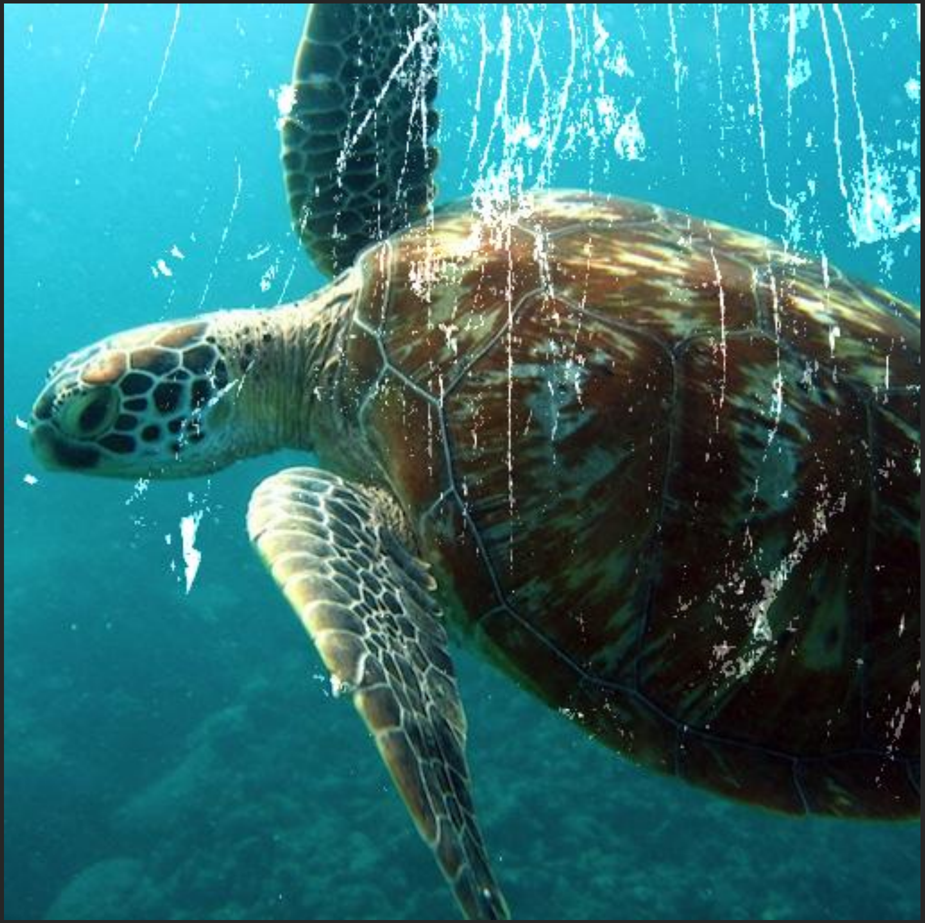
Complete the matrix



5	3			7				
6			1	9	5			
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6					2	8	
			4	1	9			5
				8			7	9

how: ?

Complete the image



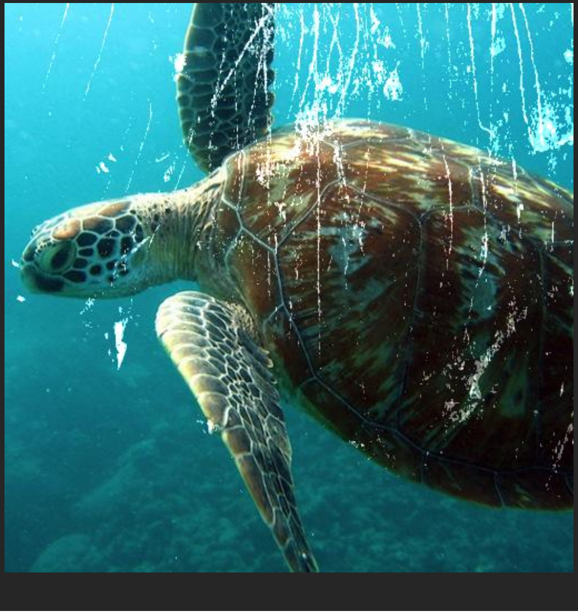
Model ?

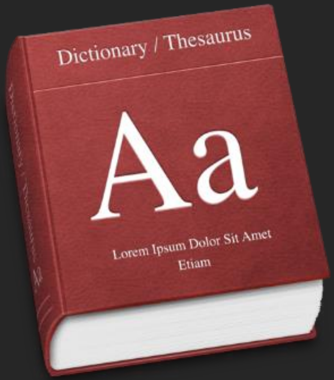
Dictionary of visual words

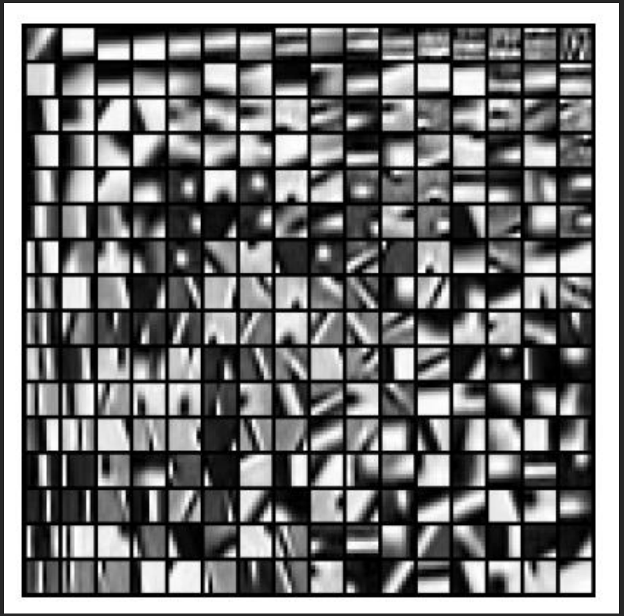
I cnt blv I m bl t rd ths sntnc.

Shrlck s th vc f th drgn

Hy, I m slvng n ndr-dtrmnd lnr systm.







Dictionary of visual words

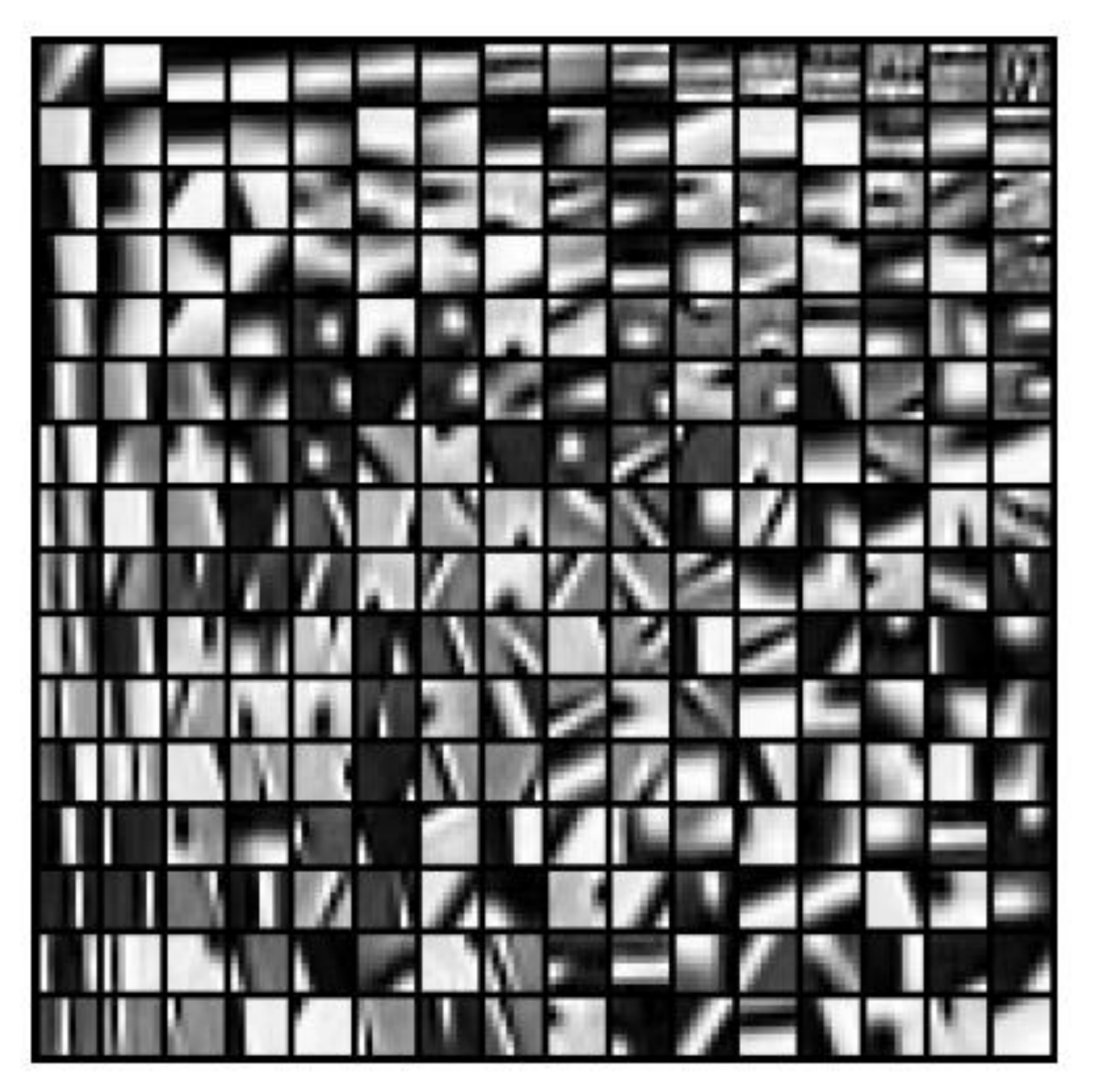


Image credit
Graeme Pope

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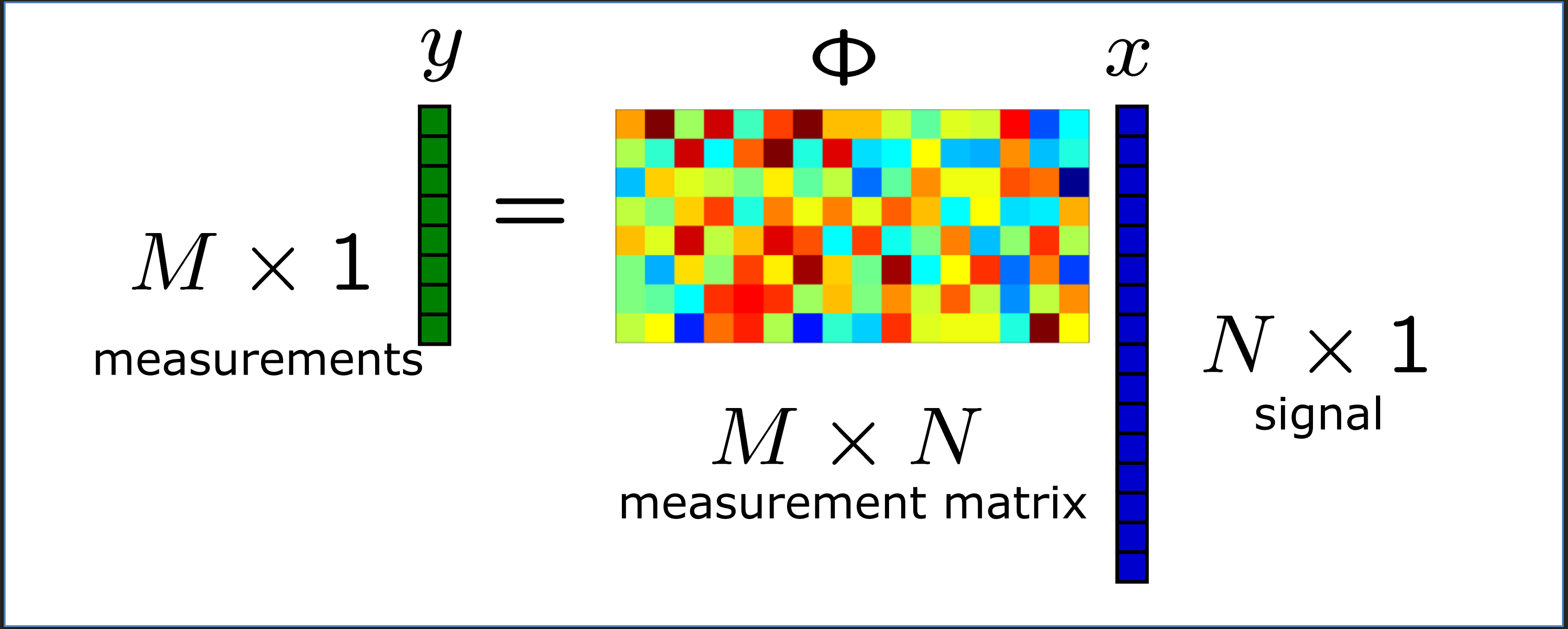
Graeme Pope

Result
Studer, Baraniuk, ACHA 2012

Compressive Sensing



$y = \Phi x$

$M \times 1$ measurements

$M \times N$ measurement matrix

$N \times 1$ signal

A toolset to solve under-determined systems by exploiting additional structure/models on the signal we are trying to recover.

Key Theoretical Ideas in Compressive Sensing

Linear Inverse Problems

- Many classic problems in computer can be posed as linear inverse problems

- Notation
  - **Signal** of interest $x \in \mathbb{R}^N$
  - **Observations** $y \in \mathbb{R}^M$
  - Measurement model $y = \Phi x + e$

measurement matrix

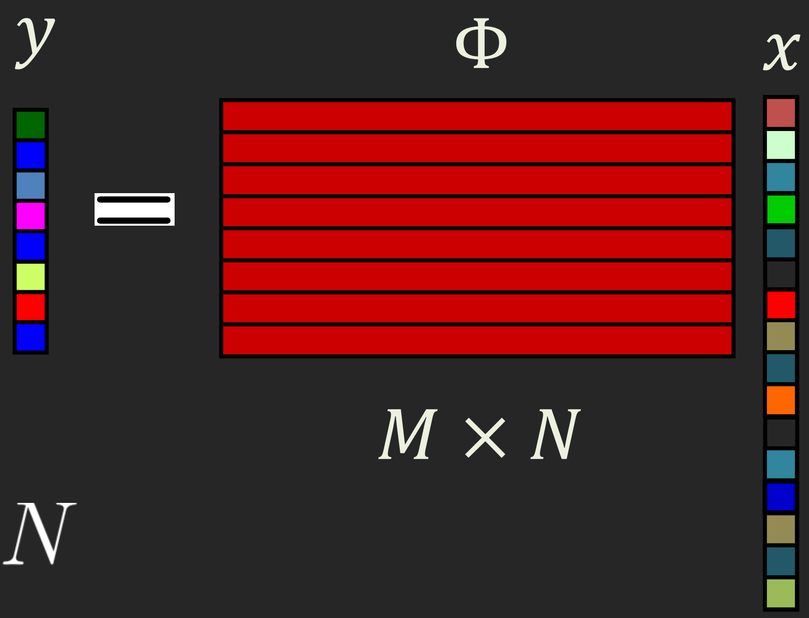
measurement noise

- Problem definition: given y, recover x

Linear Inverse Problems

$$y = \Phi x$$

M measurements



N-dim signal

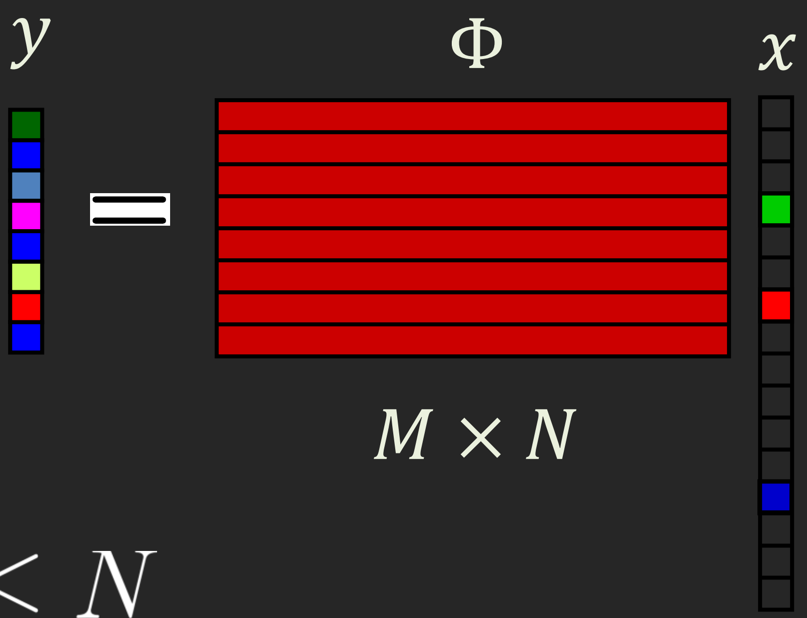
$$M < N$$

Measurement matrix has a *(N-M)* dimensional **null-space**

Solution is no longer **unique**

Sparse Signals

$$y = \Phi x$$



M measurements

N-dim signal

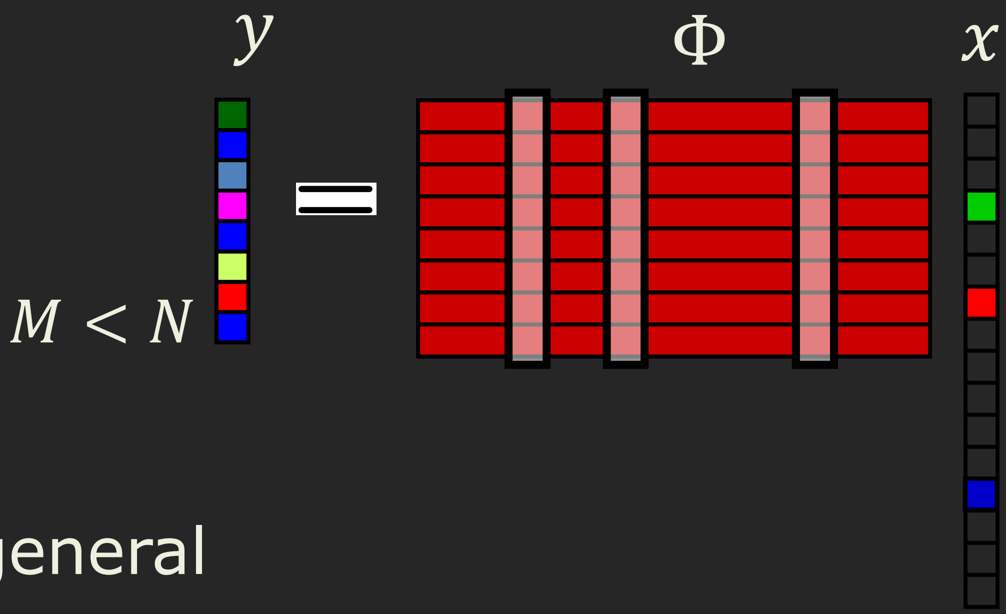
K nonzero entries

$$K < M \leq N$$

How Can It Work?

- Matrix Φ
  not full rank…

  … and so
  loses information in general



- **But we are only interested in recovering sparse signals**

Two Key Ideas

- Idea 1 --- An invertible mapping on the space of sparse signals!!!

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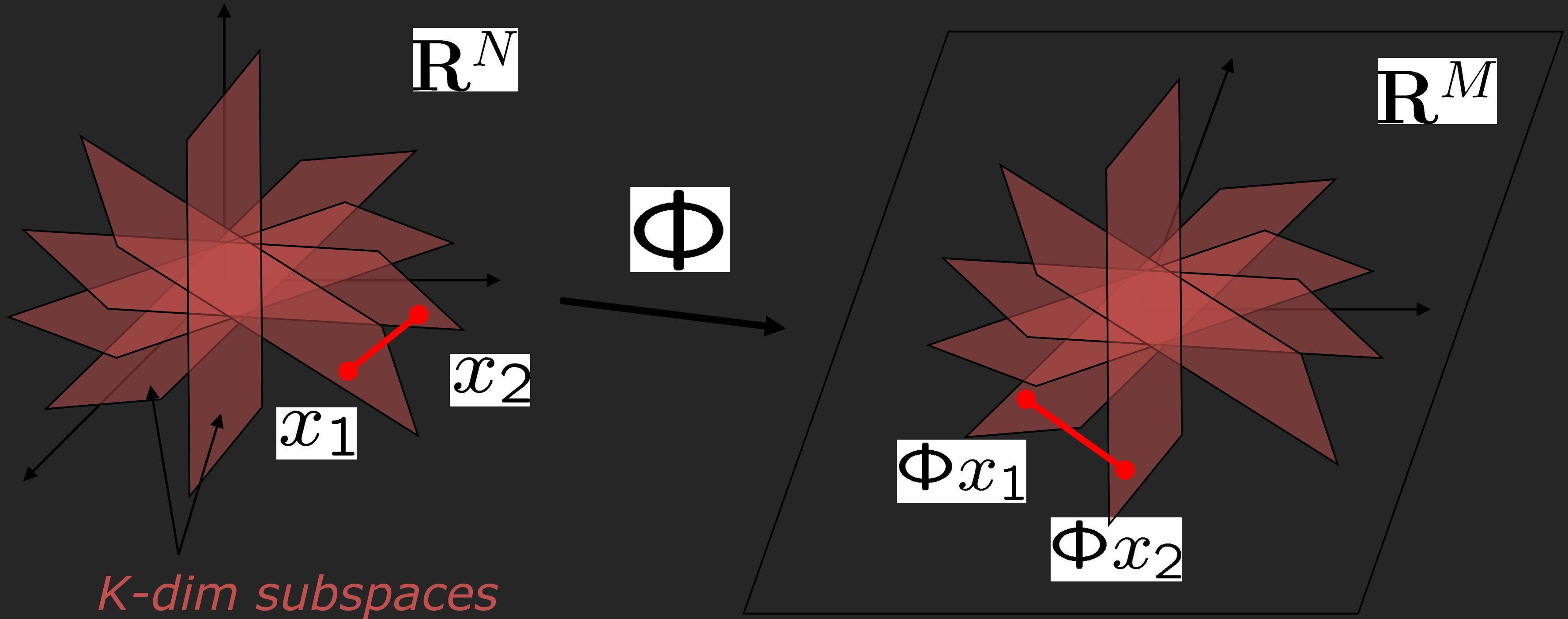
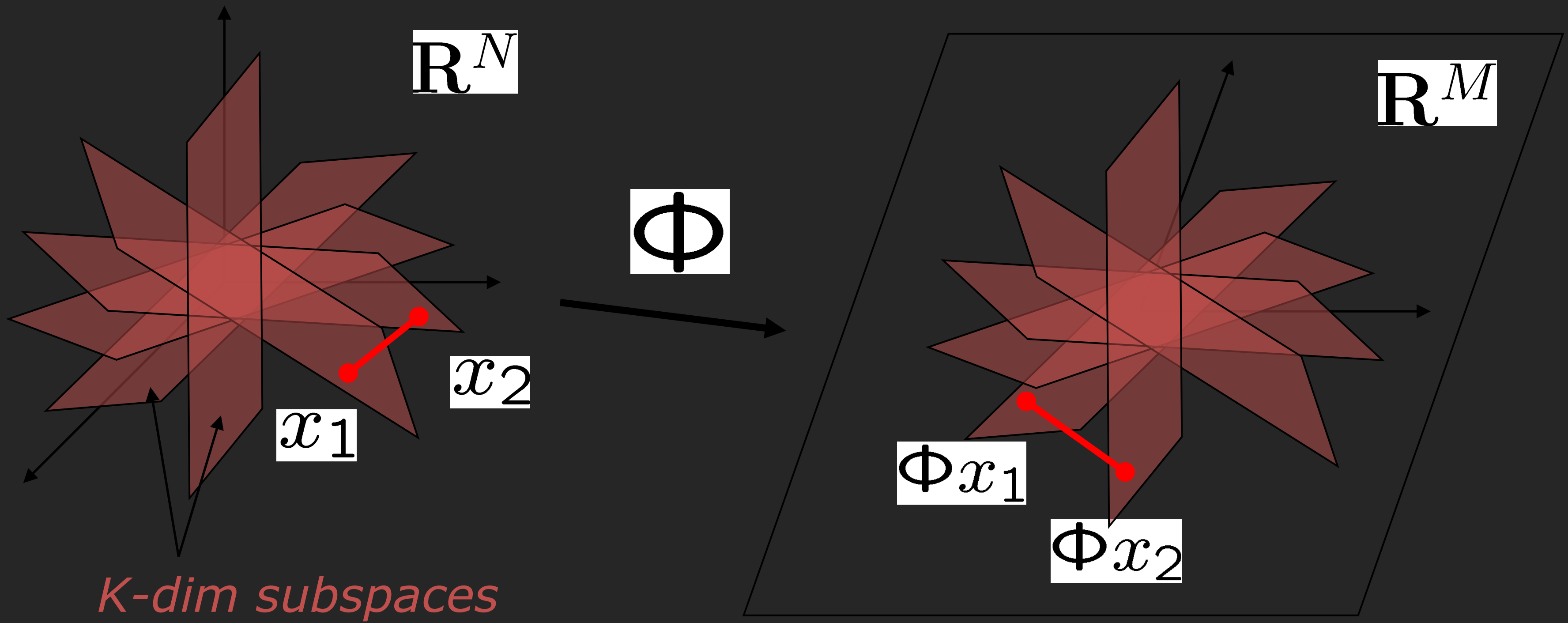
- Design Φ such that no two sparse signals x_1 and x_2 such that $\Phi x_1 = \Phi \mathrm{x}_2$



Restricted Isometry Property (RIP)

- RIP of order *2K* implies: for all *K*-sparse x_1 and x_2

$$(1-\delta_{2K}) \leq \frac{\|\Phi x_1 - \Phi x_2\|_2^2}{\|x_1 - x_2\|_2^2} \leq (1+\delta_{2K})$$



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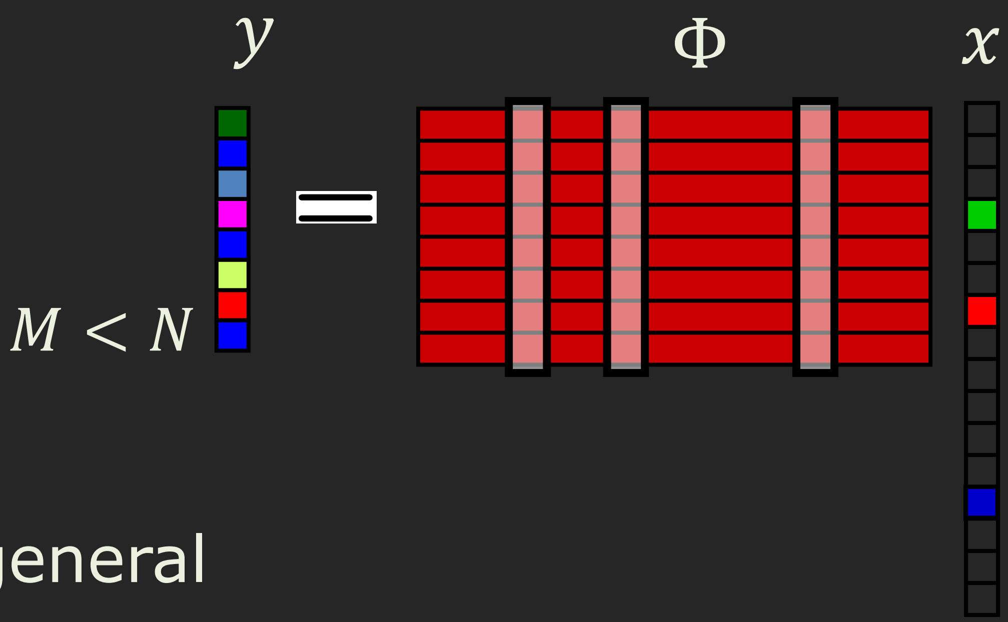

- **Design** Φ so that each of its *M*x2*K* submatrices are full rank (RIP)

How Can It Work?

- Matrix Φ
  not full rank…

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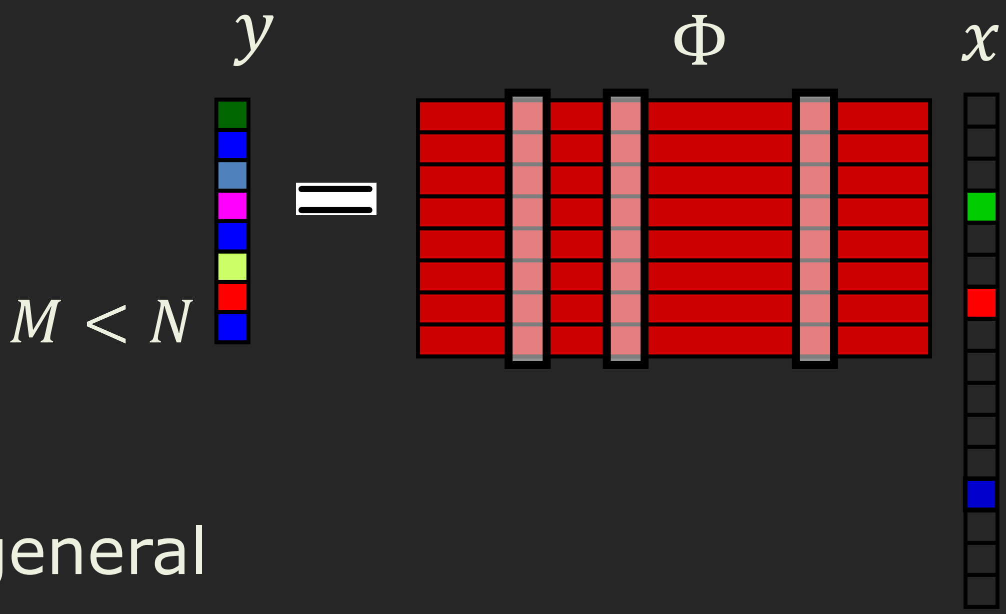


- **Design** Φ so that each of its *M*x2*K* submatrices are full rank (RIP)

- Random measurements provide RIP with

$$M \sim K \log(N/K)$$

Two Key Ideas

- Idea 1 --- An invertible mapping on the space of sparse signals!!!

- Idea 2 --- Recovery of signals: use sparse priors!

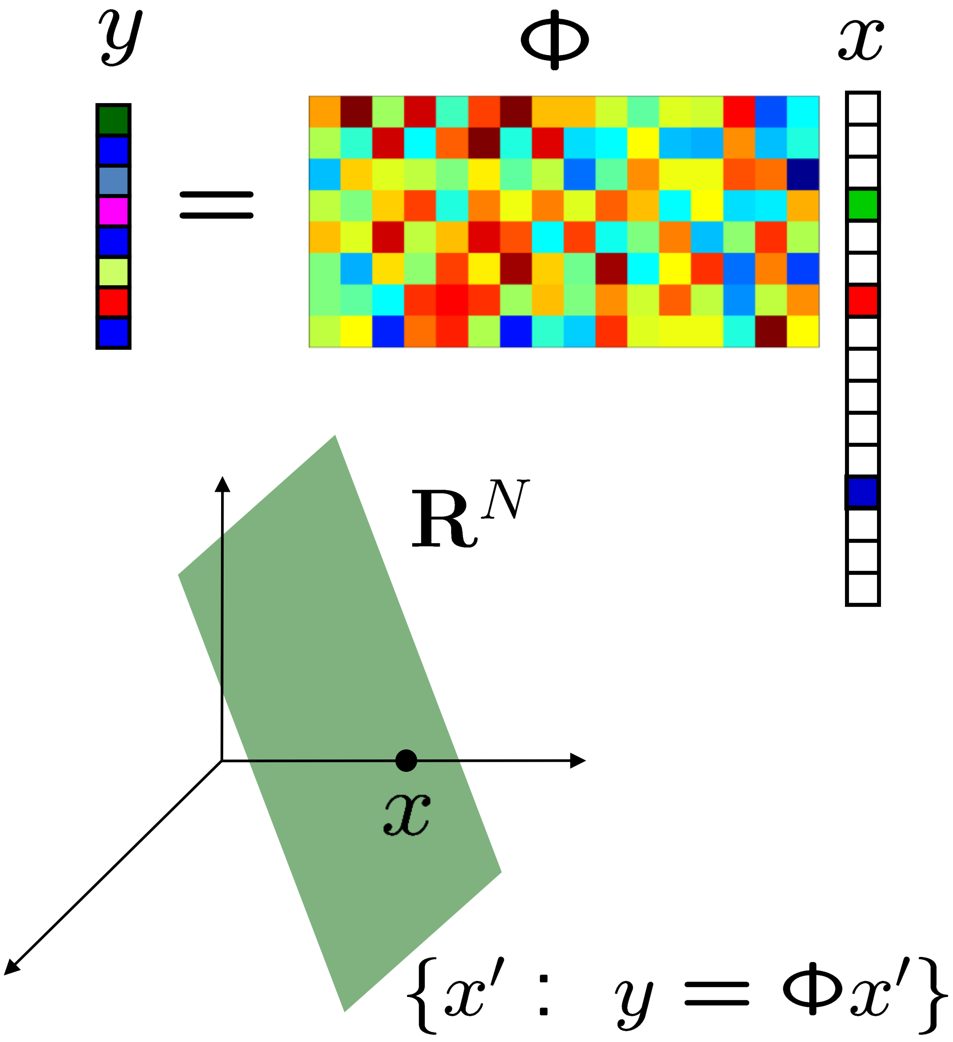
CS Signal Recovery

- Random projection Φ not full rank

- Recovery problem: given $y = \Phi x$ find x

- **Null space**

- Search in null space for the “sparsest” x



$$\{x' : \; y = \Phi x'\}$$

(N-M)-dim hyperplane
*at **random angle***

ℓ_1 Signal Recovery

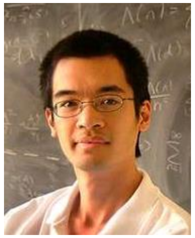
- Recovery: given $y = \Phi x$
  (ill-posed inverse problem) find x (sparse)

- Optimization: $\widehat{x} = \arg\min_{y=\Phi x} \|x\|_1$

- **Convexify** the ℓ_0 optimization

Candes

Romberg



Tao

Donoho

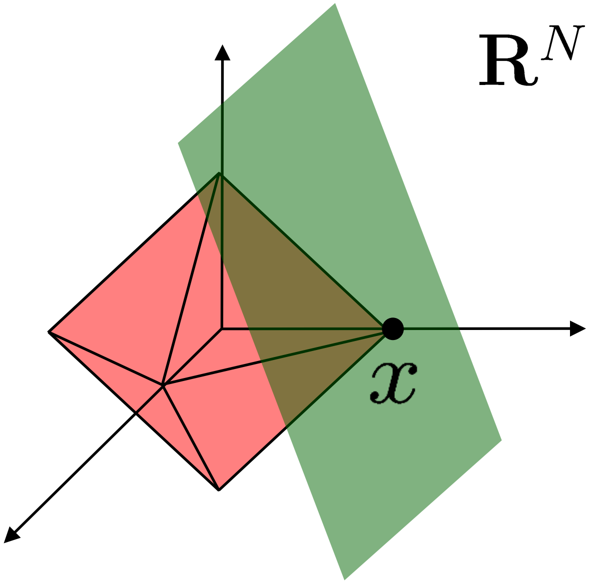
ℓ_1 Signal Recovery

- Recovery: given $y = \Phi x$
  (ill-posed inverse problem) find x (sparse)

- Optimization: $\widehat{x} = \arg\min_{y=\Phi x} \|x\|_1$

- **Convexify** the ℓ_0 optimization

- **Polynomial time** alg
  (linear programming)



Compressive Sensing

Let. $y = \Phi x_0 + e$

$$\widehat{x} = \arg\min_x \|x\|_1 \quad s.t. \quad \|y - \Phi x\|_2 \leq \|e\|$$

If Φ satisfies RIP with $\delta_{2K} \leq \sqrt{2} - 1$,

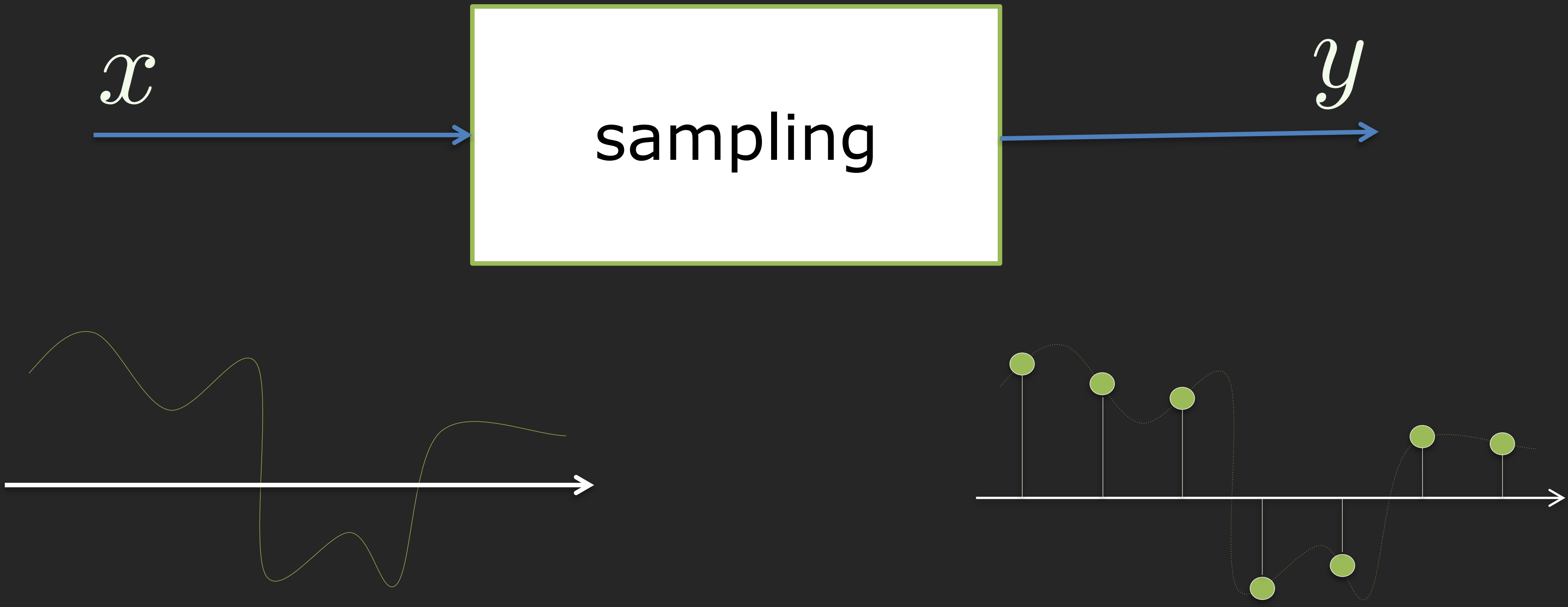
Then

$$\|\hat{x} - x_0\|_1 \leq C_1 \|e\|_2 + C_2 \|x_0 - x_{0,K}\|_2 / \sqrt{K}$$

Best K-sparse approximation

modern sensors are linear systems!!!

Sampling



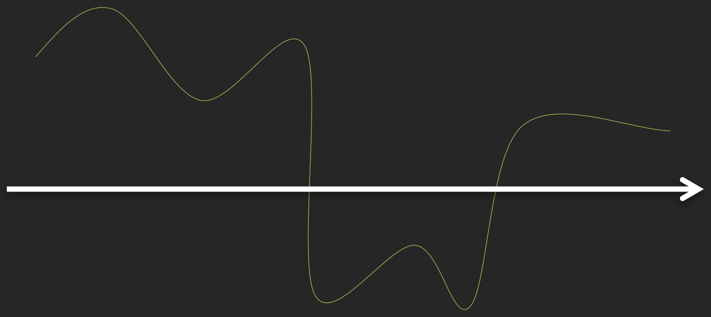
x

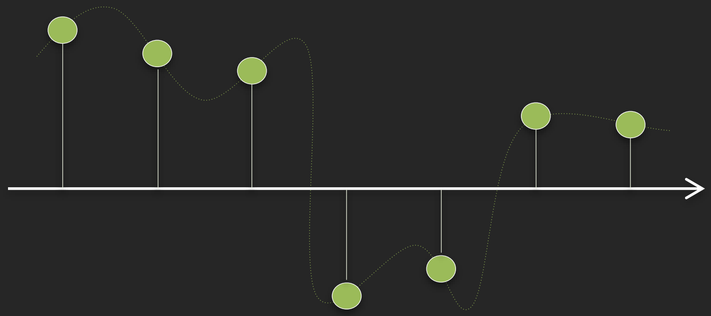
sampling

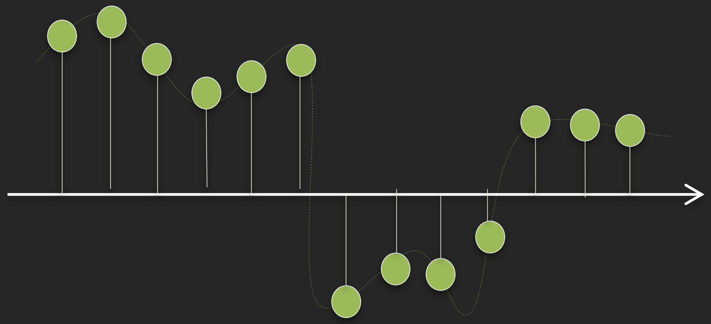
y

Can we recover the analog signal from its discrete time samples ?

Nyquist Theorem







An analog signal can be reconstructed perfectly from discrete samples *provided you sample it densely*.

The Nyquist Recipe

sample faster

sample denser

the more you sample,
the more detail is preserved

What you must learn is that these rules are no different than the rules of a computer system. Some of them can be bent. Others can be broken.

The Nyquist Recipe

sample faster

sample denser

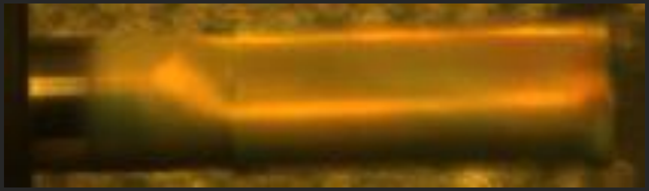
the more you sample,
the more detail is preserved

But what happens if you do not follow the Nyquist recipe ?

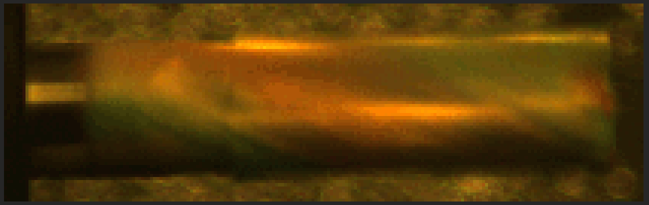
Breaking resolution barriers

- Observing a 2000 fps spinning tool with a 25 fps camera

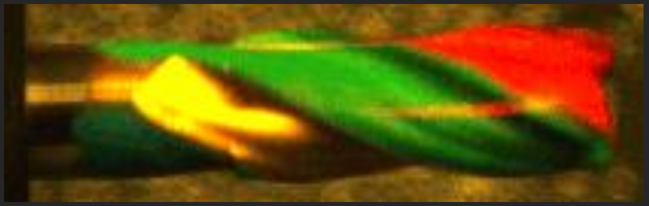
Normal Video: 25fps



Compressively obtained video: 25fps



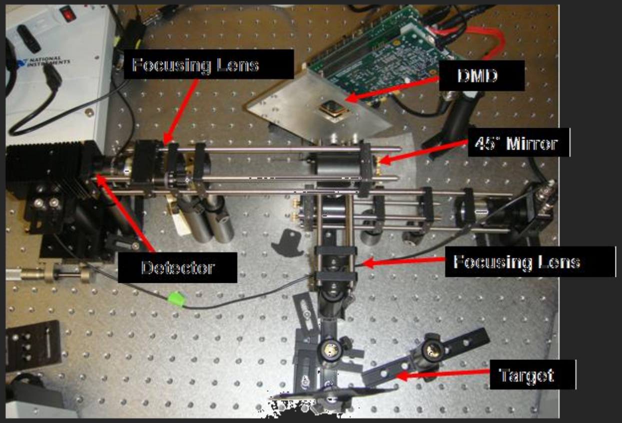
Recovered Video: 2000fps



Slide/Image credit: Reddy et al. 2011

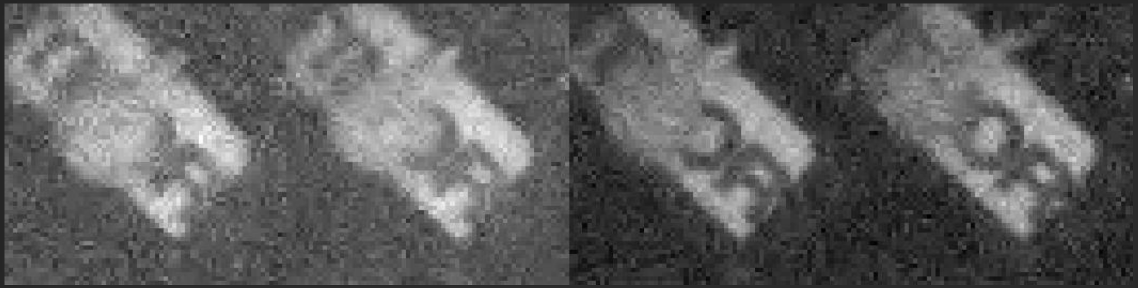
Compressive Sensing

Use of **motion flow-models** in the context of compressive video recovery

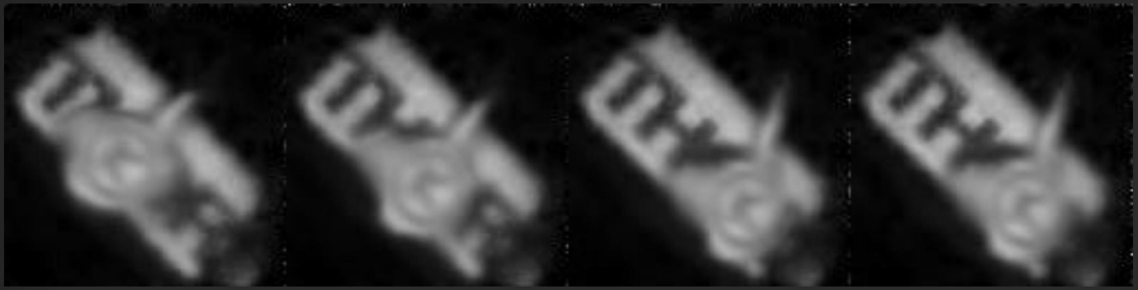


single pixel camera

128x128 images sensed at 61x comp.



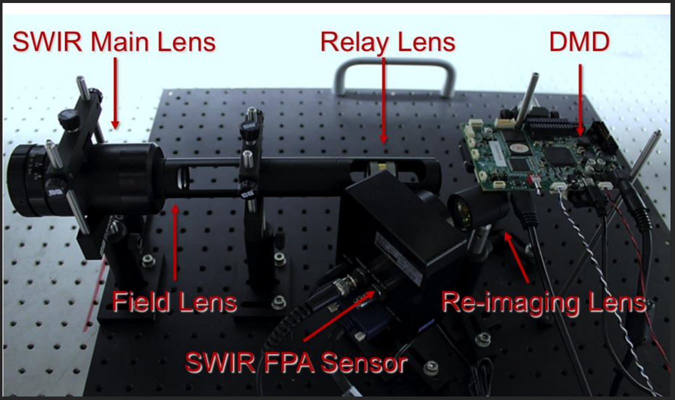
Naïve frame-to-frame recovery



CS-MUVI at 61x compression

Sankaranarayanan et al. ICCP 2012, SIAM J. Imaging Sciences, 2015*

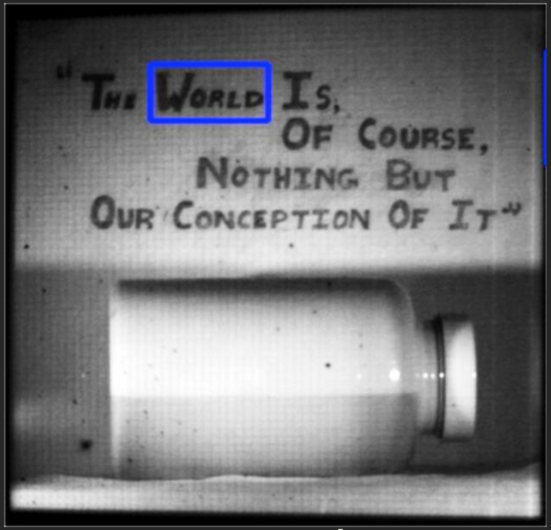
Compressive Imaging Architectures



Scalable imaging architectures that deliver videos at **mega-pixel resolutions** in infrared

visible image

SWIR image



A mega-pixel image obtained from a 64x64 pixel array sensor

Chen et al. CVPR 2015, Wang et al. ICCP 2015