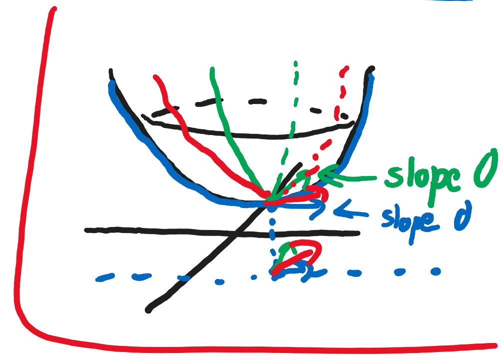
Calculus of Variations for Minimal Surfaces

First, a 1D analogue:

$f:[a,b]\to\mathbb{R}$ minimizing $\quad E(f)=\int_a^b (f')^2\,dx$

$$\boxed{\text{satisfies } \left.\frac{\partial E(f+\lambda u)}{\partial\lambda}\right|_{\lambda=0}=0 \quad \forall \text{ "test fns" } u:[a,b] \text{ w/ } \begin{matrix}u(a)=0\\ u(b)=0\end{matrix}}$$

(variations)

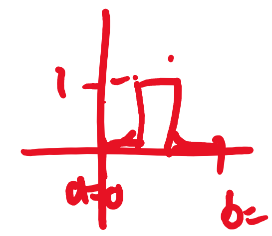


$$E(f+\lambda u)=\int_a^b (f'+\lambda u')^2\,dx$$
$$=\int_a^b (f')^2+2\lambda f'u'+\lambda^2(u')^2\,dx$$

$$\left.\frac{\partial E(f+\lambda u)}{\partial\lambda}\right|_{\lambda=0}=2\int_a^b f'u'\,dx$$

(int by parts) $$=2\left[(f'u]_a^b-\int_a^b f''u\,dx\right]$$

$$=-2\int_a^b f''u\,dx=0 \quad \underline{\forall u} \quad \Rightarrow f''=0$$



Surface case:

$$E(\vec{x})=\iint \|\vec{x}_{,u}\|^2+\|\vec{x}_{,v}\|^2\,du\,dv=\iint \|\nabla x\|^2+\|\nabla y\|^2+\|\nabla z\|^2\,du\,dv$$

$$\left(\vec{x}=\begin{pmatrix}x\\y\\z\end{pmatrix}\right)$$

$$\begin{pmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v}\end{pmatrix}$$

Int-by-parts + Divergence Thm:

$$\int_\Omega \nabla u\cdot\vec{V}=\int_{\Gamma=\partial\Omega} u(\vec{V}\cdot\hat{n})-\int_\Omega u(\nabla\cdot\vec{V})$$

Consider just one component:

$E(\vec{x})=\iint\|\nabla x\|^2\,du\,dv$ $\qquad$ variation: $\sigma=\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix}$ $\quad \sigma(\partial\Omega)=0$

$$E(\vec{x}+\lambda\vec{\sigma})=\iint\|\nabla x+\lambda\nabla\alpha\|^2\,du\,dv$$
$$=\iint\|\nabla x\|^2+2\lambda\langle\nabla x,\nabla\alpha\rangle+\lambda^2\|\nabla\alpha\|^2\,du\,dv$$

$$\left.\frac{\partial E(\vec{x}+\lambda\vec{\sigma})}{\partial\lambda}\right|_{\lambda=0}=2\iint(\nabla x)\cdot(\nabla\alpha)\,du\,dv$$

$\left(u=\alpha \quad \vec{V}=\nabla x \quad \text{int-by-parts}\right)$

$$=2\left[\int_\Gamma\alpha(\nabla x\cdot\hat{n})-\iint\alpha(\nabla\cdot\nabla x)\,du\,dv)\right]$$

$$=-2\iint(\Delta x)\,\alpha\,du\,dv$$

Vanishes for all α $\Rightarrow \Delta x=0$ in interior

Similar argument holds for y,z cpts

$\Delta y=\Delta z=0$ in interior