Digital 3D Geometry Processing
Exercise 3 – Curves

February 8, 2024

Note

Hand in a .zip compressed file renamed to `Exercise`n`-GroupMemberNames.zip` where n is the number of the current exercise sheet. It should contain:

- **Only** the files you changed (headers and source). It is up to you to make sure that all files that you have changed are in the zip.
- A `readme.txt` file containing a description on how you solved each exercise (use the same numbers and titles) and the encountered problems.
- Other files that are required by your `readme.txt` file. For example, if you mention some screenshot images in `readme.txt`, these images need to be submitted too.
- Submit your solutions to Gradescope before the submission deadline.

1 Theory Exercise (7 pt)

1.1 Curvature of Curves (3 pt)

Match each curve from Figure 1 with the corresponding curvature function:

$$k_1(s) = \frac{s^2 - 1}{s^2 + 1}, \quad k_2(s) = s, \quad k_3(s) = s^3 - 4s, \quad k_4(s) = \sin(s)s. \tag{1}$$

Write your solutions in a file named `TheoryExercise.pdf`. If, for example, the curvature function $k_1(s)$ corresponds to the curve **a**, write **1-a** as your solution. **Briefly explain your answers.**

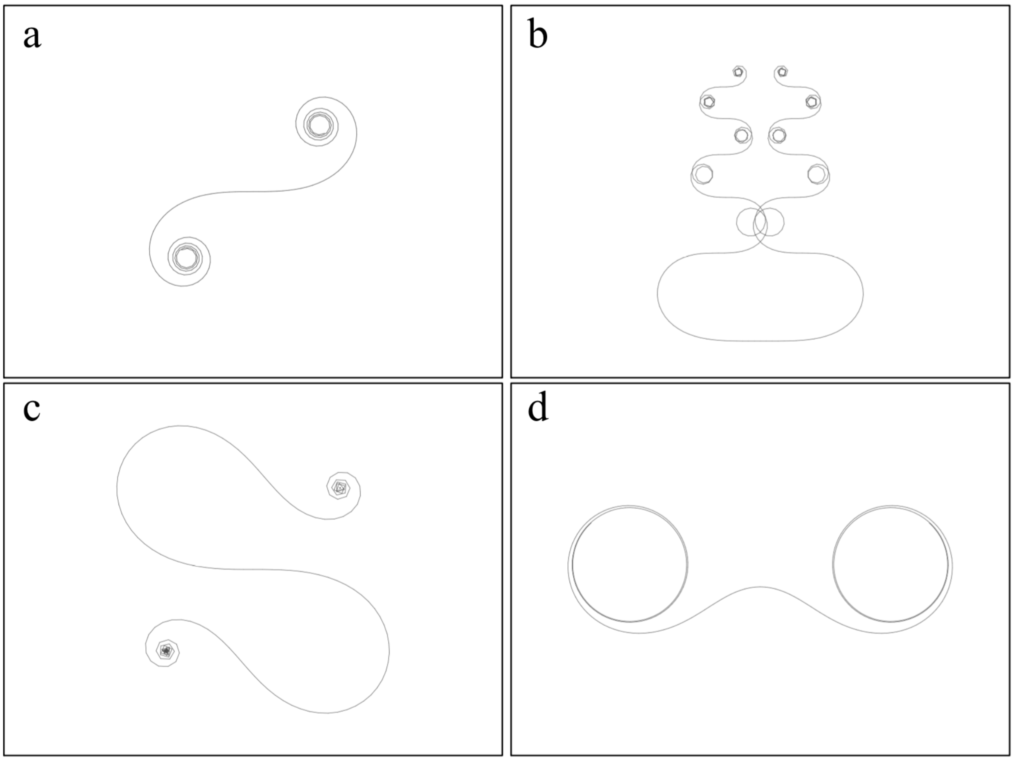

Figure 1: Curves reconstructed from given curvature functions $k_i(s)$.

Sketch (approximately) a curve that has curvature function:

$$k_5(s) = \frac{e^s - e^{-s}}{e^s + e^{-s}}$$

(**Hint:** graphing may help, or looking up "hyperbolic trigonometric functions"). Take an image of your curve and include it as `k5.png/jpg/whatever` in the zip.

1.2 Surfaces Area (2 pt)

King Archimedes wants to renovate his palace. The most striking structure is a spherical half-dome of 20m in diameter that covers the great hall. The king wants to cover this dome in a layer of pure gold. He has decided to split the work into two parts, each one covering a vertical slice of the dome of the same height (see Figure 2). For each part he hires different people and gives them 700kg of gold. The task is to cover the surface of one vertical slice with a layer of gold of 0.1mm thickness (the thickness is small enough so that it can be ignored).The amount of gold that is left over is the salary for doing the job. Which slice should you pick if you want to make the most profit? Explain your answer in `TheoryExercise.pdf`.
How does your answer change when you have n slices instead of just two?

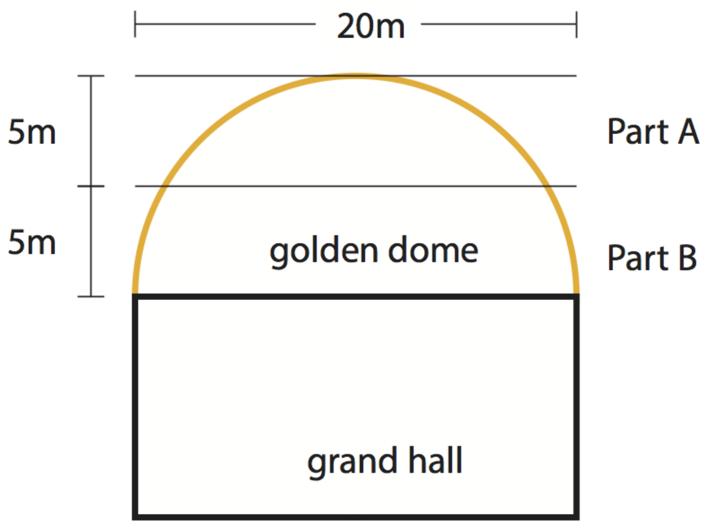

Figure 2: Sketch of King Archimedes dome.

1.2 Jacobian and First Fundamental Form (2 pt)

Consider the parameterized surface given by the graph of the function $f(u,v) = -u^2 + v^2$.

$$\mathbf{x}(u,v) = \begin{bmatrix} u \\ v \\ -u^2 + v^2 \end{bmatrix}$$

a. What is the Jacobian $J(u,v)$ of the parameterized surface?

b. What is the first fundamental form $\mathbf{I}(u,v)$?

c. Using $\mathbf{I}$, find the angle between the two curves on the surface given by $\mathbf{x}(\gamma_1(t))$ and $\mathbf{x}(\gamma_2(s))$ where:

$$\gamma_1(t) = (t,1)$$
$$\gamma_2(t) = (s,s-1)$$

d. Write out an integral (no need to evaluate) for the surface area of this graph over the region $-1 \leq u \leq 1, -1 \leq v \leq 1$.

2 Coding Exercise (6 pt)

2.1 The Task

Download `Plugin-DGPExercises.zip` from ILIAS. Extract and replace the files in **Plugin-DGPExercise** with the new ones. For the exercise 4 you will need to fill in the missing code in the file `CurveSmoothing.cc`.
Given a curve in $\mathbb{R}^2$ as a closed polyline with vertices $\{V_i\}_{i=1}^M$, where $V_1 = V_M$, implement the following two examples of curve smoothing:

- Move every vertex towards the centroid of the two neighbors. More specifically, every vertex V_i should shift to vertex V_i' according to

$$V_i' = (1-\varepsilon)V_i + \varepsilon \frac{V_{i-1} + V_{i+1}}{2} \tag{2}$$

  where ε is a small time step (you can experiment with different values for the time step). Iterate this procedure. After each iteration uniformly scale the curve to its original length around its current centroid.
  Implement this smoothing by filling in the `laplacian_smoothing()` function in the file `CurveSmoothing.cc`. In the spinbox of the plugin, you can specify the iteration times. By pressing the corresponding button, the function performs smoothing.

- Move every vertex towards the center of the osculating circle. Consider an osculating circle at vertex V_i as a circumscribed circle O of a triangle defined with vertices V_{i-1}, V_i and V_{i+1}. This can be done by shifting every vertex V_i to vertex V_i' according to

$$V_i' = V_i + \varepsilon \frac{C - V_i}{\|C - V_i\|^2} \tag{3}$$

  where ε is a small time step and C is the center of the circumscribed circle O. Iterate this procedure. After each iteration uniformly scale the curve to its original length around its current centroid.

  Implement this smoothing by filling in the `osculating_circle()` function in the file `CurveSmoothing.cc`. In the spinbox of the plugin, you can specify the iteration times.. By pressing the corresponding button, the function performs smoothing.

To run the smoothing algorithms on different examples, generate curves within the `Plugin-DGPExercise`.