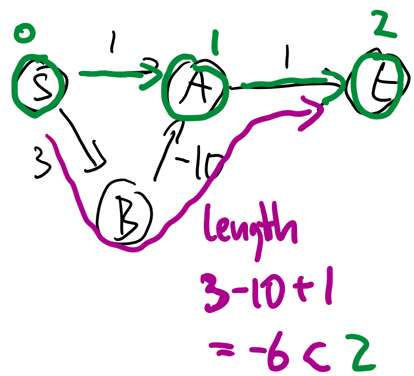
09 Bellman Ford/Knapsack

Thursday, February 6, 2025 13:55

Shortest path with negative edge costs

Dijkstra can fail



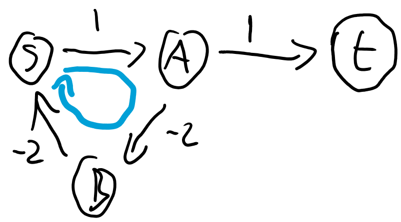
Queue

~~A 1~~

B 3

~~t 2~~

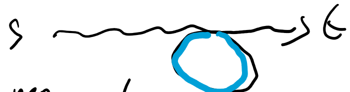
No shortest way to go from s to t



it's always better to do one more s->A->B->s cycle because it reduces the length by 3

Today: Algorithm that works if the graph has no cycle of negative cost.

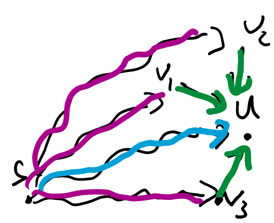
Bellman-Ford



Observation: If there are no neg. cycles then the shortest paths do not have cycles

=> a shortest path uses at most $|V|-1$ edges
visit at most $|V|$ vertices

$T[u, i]$ = length of shortest path from s to u that uses at most i edges



$$T[u, i+1] = \min\left(T[u, i],\ \min_{(v,u)\in E}\left(T[v, i] + c_{vu}\right)\right)$$

$$T[u, 1] = \begin{cases} c_{su} & \text{if } (s,u)\in E \\ \infty & \text{if } (s,u)\notin E \end{cases} \quad \forall\, v \in V$$

$$T[u,1] = \begin{cases} c_{su} & \text{if } (s,u) \in E \\ \infty & \text{if } (s,u) \notin E \end{cases} \quad \forall\, v \in V$$

BellmanFord$(G=(V,E),\ s)$

for $u \in V$
  $T[u,1] = c_{su}$ if $(s,u) \in E$
  $T[u,1] = \infty$ if $(s,u) \notin E$
$T[s,1] = 0$

Base case $O(|V|)$

Break 2:48

for $i = 2 \ldots |V|-1$ $O(|V|)$
  for $u \in V$
    $T[u,i] = \min\left(T[u,i-1],\ \min\limits_{(v,u)\in E}\left(T[v,i-1] + c_{vu}\right)\right)$

Recursion $O(|E|)$

total $O(|V| \cdot |E|)$

distance = [ ]
for $v \in V$
  distance$[v] = T[v, |V|-1]$
return distance

output

Example

$$\begin{tikzcd} S \arrow[r, "10"] \arrow[d, "1"'] & B \\ C \arrow[r, "2"'] & D \arrow[u, "3"'] \end{tikzcd}$$

T	S	B	C	D
i=1	0	10	1	∞
i=2	0	10	1	3
i=3	0	6	1	3

$$T[u,i+1] = \min\Big(\underbrace{T[u,i]}_{10},\ \min_{(v,u)\in E}\big(T[v,i] + c_{vu}\big)\Big)$$

3 + 3 (from D)

0 + 10 (from S)

BellmanFord$(G=(V,E),\ s)$

for $u \in V$
  $T[u,1] = c_{su}$ if $(s,u) \in E$
  $T[u,1] = \infty$ if $(s,u) \notin E$
$T[s,1] = 0$
for $i = 2 \ldots |V|-1$

$T[s,1] = 0$

for $i = 2, \dots, |V|-1$

  for $u \in V$

    $T[u,i] = \min\left(T[u,i-1],\ \min_{(v,u)\in E}\left(T[v,i-1] + c_{vu}\right)\right)$

* make $|V|$ extra iterations

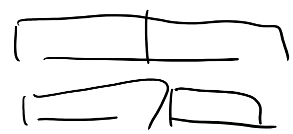
for $u \in V$

  if $\left(\min_{(v,u)\in E}\left(T[v,|V|-1] + c_{vu}\right)\right) < T[u,|V|-1]$

    return "there is neg. cycle"

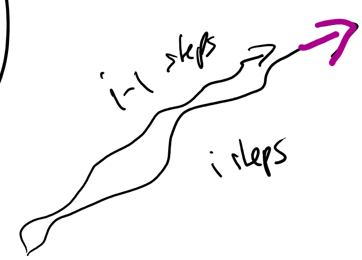
return "no neg cycle"

Dynamic Programming



- Describe problem/solution recursively

  (Unlike D&C do not split problem in half. Instead, cut off a small piece of the input or the solution.)



- Find base cases
- Fill table via for-loops
- Which cells of table contain solution?

Knapsack: bag of size $B \in \mathbb{N}$

n items $V[i]$ is value of i-th item $V[i], S[i] \in \mathbb{N}$

$S[i]$ is the size of ith item. $S[i] \leq B$

Find $S \subseteq \{1 \dots n\}$ with $\sum_{i \in S} S[i] \leq B$

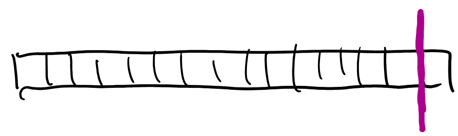
Find $S \subseteq \{1 \dots n\}$ with $\sum_{i \in S} s[i] \leq B$

such that the value $\sum_{i \in S} v[i]$ is as large as possible.

$T[i, b]$ = max value possible if we have b space left and only the first i items available as option

$$T[i, b] = \max\left(T[i-1, b],\ T[i-1, b - s[i]] + v[i]\right)$$

do not pack i-th item



Base case

$T[i, 0] = 0 \quad \forall i = 0 \dots n$

$T[i, b] = -\infty \quad \forall b < 0$

$T[0, b] = 0 \quad \forall b\ 0 \dots B$

for $i = 1 \dots n$
  for $b = 1 \dots B$
    $T[i, b] = \max(T[i-1, b], T[i-1, b - s[i]] + v[i])$

return $T[n, B]$

$O(nB)$

"pseudo polynomial"

polynomial in <u>value</u> of B

Array length n $\quad v[1 \ldots n]$

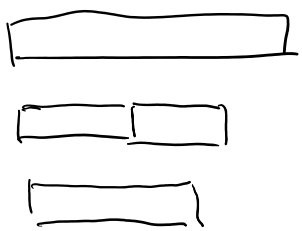
$S[1 \ldots n]$

$B \leftarrow \log B$ bit

Input size B $\quad n+k \quad k = \log B$

time $O(nB) = O(n2^k)$

Next week polynomial time for $(1 \pm \varepsilon)$-approximation



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