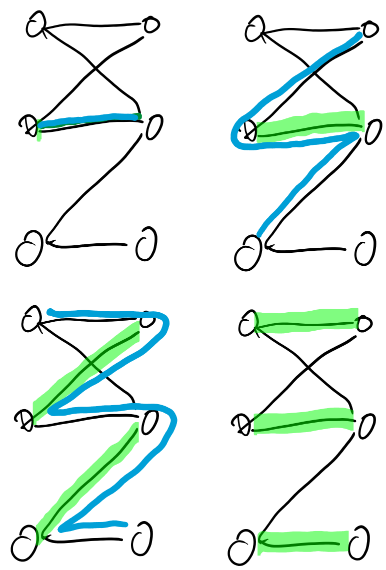
12 Bip. Matching (cont.), Max Flow, Min Cut

Tuesday, February 18, 2025 13:53

Bipartite Matching



Input (G)

$M = \emptyset$

while $\exists$ augmenting path P in G

$M \leftarrow M \Delta P$ // augment

// $(M \cup P) \setminus (M \cap P)$

return M

#iteration $\leq \frac{|V|}{2}$

time complexity $O(|V| \cdot \text{time to find path})$

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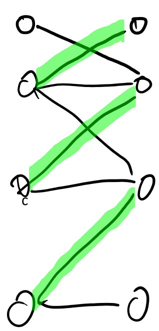
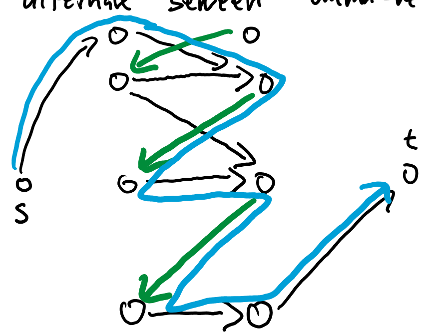
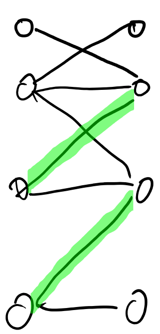
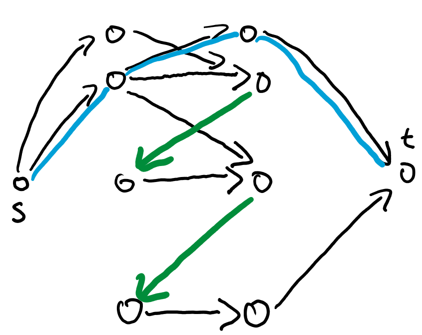
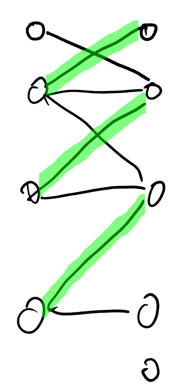
augmenting path
- starts at unmatched vertex
- ends at unmatched vertex
- ✓ alternate between unmatched/matched edge











Given G, M

orient unmatched edges left -> right

matched edges right -> left

add vertices s, t

connect $s \to v$ $\forall$ unmatched v on left

$O(|E| + |V|)$

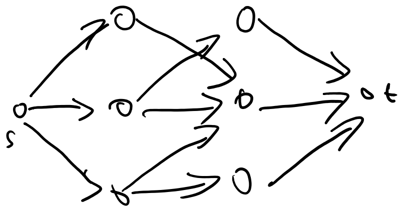
$O(|E|)$

=> Find max matching

add vertices s, t
connect $s \to v$ $\forall$ unmatched v on left
$u \to t$ $\forall$ unmatched u on right

$\Rightarrow$ Find max matching in $O(|V| \cdot |E|)$

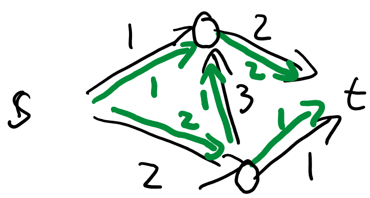
Find path from s to t.



Hopcroft Karp $O(\sqrt{|V|}\,|E|)$

Chen Peng $|E|^{1+o(1)}$ $|E|^{1.000001}$ almost $|E|$
Sachdeva Kyng Gutenberg Liu 2022

Max Flow



directed graph $G = (V, E)$
edge capacities $\mathbb{N}$
How much flow can we send from s to t?

- everything that enters a vertex must also leave.
- follow edge direction
- can send at most the capacity along each edge

$f \in \mathbb{R}^E$
f_e = flow on edge e

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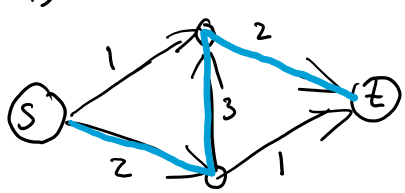
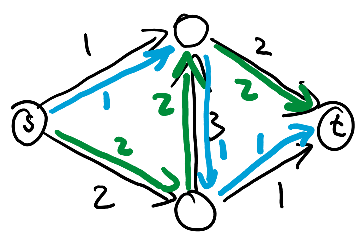


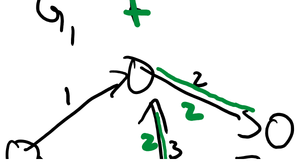


Given graph $G = (V, E)$ and flow f residual graph G_f

$G_f = (V, E')$

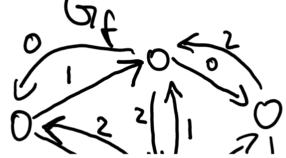
for each $(u,v) \in E$
add (u,v) with capacity $c_{uv} - f_{uv}$, skip if $c_{uv} - f_{uv} = 0$
add (v,u) with capacity f_{uv}, skip if $f_{uv} = 0$

G, f

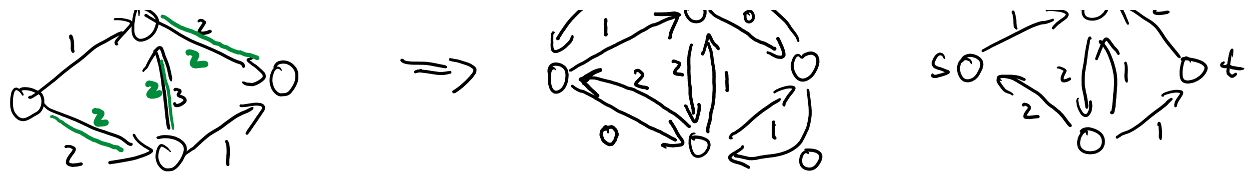


$\Rightarrow$

G_f



G_f



Algorithm:

Input (G, s, t)

$f_e = 0 \quad \forall e \in E$

while there exists st path in G_f

- let p be the smallest capacity on the path in G_f
- for e in path
  - $f_e = f_e + p$ if e existed in G
  - $f_e = f_e - p$ if flipped e existed in G

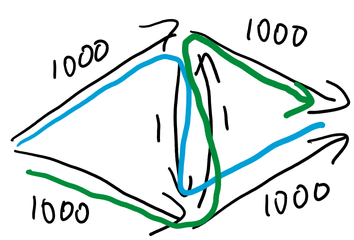
return f // maximum st flow

\# iterations $\leq$ size(f)

how much flow is sent.

time: $O(F \cdot |E|)$

F size of max flow



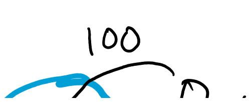
Is f a max st flow when the algorithm terminates?

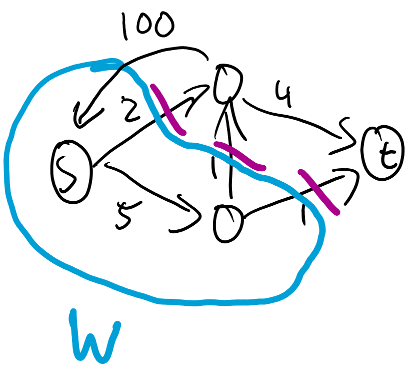
If there is no path in G_f, does that imply f is max flow?

Min st cut problem:

Given G, s, t — find set of edges such that removing the edges (cutting) there is no path left from s to t.

Cost of cut is sum of capacities





cost of cut is sum of capacities

cost $2+1+1=4$

equivalent $W \subseteq V$

$s \in W \quad t \notin W$

$$\delta(W) = \sum_{\substack{(u,v) \in E \\ u \in W \\ v \notin W}} c_{uv}$$