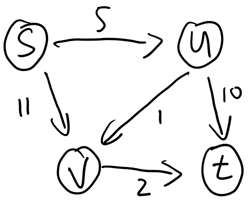
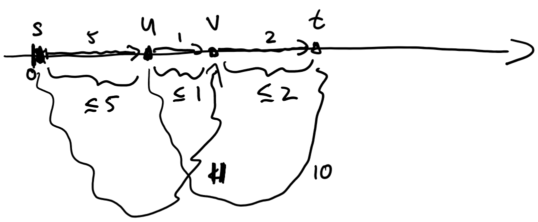


$$\max \; x_t - x_s$$

$$x_u - x_s \le 5$$
$$x_v - x_s \le 11$$
$$x_v - x_u \le 1$$
$$x_t - x_u \le 10$$
$$x_t - x_v \le 2$$



General

$G = (V, E)$ c_e

$$\max \; x_t - x_s$$

$$x_v - x_u \le c_{uv} \quad \forall (u,v) \in E$$

if we add $x_s = 0$ then for optimal solution x

x_u is distance from s to u if u is on shortest path from s to t

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Matrix representation

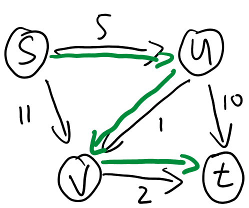
$$\max \; x_t - x_s$$

$$x_u - x_s \le 5$$
$$x_v - x_s \le 11$$
$$x_v - x_u \le 1$$
$$x_t - x_u \le 10$$
$$x_t - x_v \le 2$$

$$\boxed{\begin{aligned}&\max \; c^T x\\ &Ax \le b\end{aligned}}$$

$$\max \; (-1 \; 0 \; 0 \; +1)\, x$$

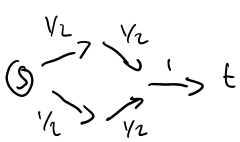
$$\begin{bmatrix} -1 & +1 & 0 & 0 \\ -1 & 0 & +1 & 0 \\ 0 & -1 & +1 & 0 \\ 0 & -1 & 0 & +1 \\ 0 & 0 & -1 & +1 \end{bmatrix} \begin{pmatrix} x_s \\ x_u \\ x_v \\ x_t \end{pmatrix} \le \begin{vmatrix} 5 \\ 11 \\ 1 \\ 10 \\ 2 \end{vmatrix}$$



$$\min \; 5 \cdot y_{su} + 11 \cdot y_{sv} + 1 \cdot y_{uv} + 10\, y_{ut} + 2\, y_{vt}$$

$$y_{su} - y_{uv} - y_{ut} = 0$$
$$y_{uv} + y_{sv} - y_{vt} = 0$$
$$\vdots$$

$y_e \in [0,1]$



$$\min \sum_{e \in E} y_e \cdot c_e$$

$$\sum_{(u,v) \in E} y_{uv} - \sum_{(v,z) \in E} y_{vz} = \begin{cases} 0 & \forall v \ne s,t \\ -1 & v = s \\ +1 & v = t \end{cases}$$

incoming to v

outgoing from v

$$y_e \ge 0$$

$$\min \; c^T y \leftarrow \sum_{e \in E} c_e \cdot y_e$$

$$\begin{array}{c|ccccc|} & su & sv & uv & ut & vt \\ \hline s & -1 & -1 & 0 & 0 & 0 \\ u & +1 & 0 & -1 & -1 & 0 \\ v & 0 & +1 & +1 & 0 & -1 \\ t & 0 & 0 & 0 & +1 & +1 \end{array} \begin{pmatrix} y_{su} \\ y_{sv} \\ y_{uv} \\ y_{ut} \\ \vdots \end{pmatrix} = \begin{vmatrix} -1 \\ 0 \\ 0 \\ 1 \end{vmatrix}$$

$$\begin{array}{c} \\ v \\ t \end{array}\begin{bmatrix} \cdots \\ 0 & +1 & +1 & 0 & -1 \\ 0 & 0 & 0 & +1 & +1 \end{bmatrix} \begin{pmatrix} \vdots \\ y_{uv} \\ y_{ut} \\ y_{vt} \end{pmatrix} \quad \begin{vmatrix} \vdots \\ 0 \\ 1 \end{vmatrix}$$

$$y \geq 0$$

---

min $c^T y$

$$\min \begin{pmatrix} 5 & 11 & 1 & 10 & 2 \end{pmatrix} y$$

$$\begin{array}{c|ccccc} & su & sv & uv & ut & vt \\ \hline s & -1 & -1 & 0 & 0 & 0 \\ u & +1 & 0 & -1 & -1 & 0 \\ v & 0 & +1 & +1 & 0 & -1 \\ t & 0 & 0 & 0 & +1 & +1 \end{array} \begin{pmatrix} y_{su} \\ y_{sv} \\ y_{uv} \\ y_{ut} \\ y_{vt} \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

M^T

$$y \geq 0$$

$$\max \begin{pmatrix} -1 & 0 & 0 & +1 \end{pmatrix} x$$

$$\begin{bmatrix} -1 & +1 & 0 & 0 \\ -1 & 0 & +1 & 0 \\ 0 & -1 & +1 & 0 \\ 0 & -1 & 0 & +1 \\ 0 & 0 & -1 & +1 \end{bmatrix} \begin{pmatrix} x_s \\ x_u \\ x_v \\ x_t \end{pmatrix} \leq \begin{pmatrix} 5 \\ 11 \\ 1 \\ 10 \\ 2 \end{pmatrix}$$

M

$$\max b^T x \qquad \longleftrightarrow \qquad \min c^T y$$

$$Mx \leq c \qquad\qquad M^T y = b$$

$$y \geq 0$$

"original" problem

Today
stretch s far from t

Past
max flow

bottleneck, the thing from preventing us to improve the solution

Today
which edges stop us from stretching further

Past
min cut, bottleneck preventing us to send more flow

$\boxed{2:54}$

max $5x_1 + 4x_2$

$2x_1 + x_2 \leq 6$

$x_1 + 2x_2 \leq 6$

$x_1 + x_2 \leq 1$

$2 \cdot (2x_1 + x_2) + 1 \cdot (x_1 + 2x_2) = 5x_1 + 4x_2$

$\leq 2 \cdot 6 + 1 \cdot 6$

$= 18$

$\Rightarrow$ no matter what $x_1\ x_2$ we pick, the cost is at most 18

valid solutions:
- $x_1 = 1 \quad x_2 = 0 \quad 5x_1 + 4x_2 = 5$
- $x_1 = 5 \quad x_2 = -4 \quad 5x_1 + 4x_2 = 9$ Optimal?

$$y_1 \cdot (2x_1 + x_2) + y_2 \cdot (x_1 + 2x_2) + y_3 (x_1 + x_2) = 5x_1 + 4x_2$$

$y \geq 0$

$\leq y_1 \cdot 6$

$\leq 6 \cdot y_1 + 6 y_2 + 1 \cdot y_3$

$y_1 \cdot \underline{(2x_1 + x_2)} + y_2 \cdot \underline{(x_1 + 2x_2)} + y_3 \cdot \underline{(x_1 + x_2)} \quad \underline{5x_1 + 4x_2}$

$\le y_1 \cdot \underline{6}$ $\qquad\qquad \le \underline{6} \cdot y_1 + \underline{6} y_2 + \underline{1} \cdot y_3$

minimize $6y_1 + 6y_2 + 1y_3$

$$(y_1\ y_2\ y_3) \begin{pmatrix} 2 & 1 \\ 1 & 2 \\ 1 & 1 \end{pmatrix} = (5\ \ 4) \qquad \Longleftrightarrow \qquad \min\ (6\ 6\ 1) \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$y \ge 0 \qquad\qquad \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$$

$$y \ge 0$$

$$\max\ (5\ \ 4) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad\qquad \min\ (6\ 6\ 1) \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \le \begin{pmatrix} 6 \\ 6 \\ 1 \end{pmatrix} \qquad\qquad \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$$

$$y \ge 0$$

P $\qquad\qquad\qquad\qquad$ D

Theorem: strong duality

$$\max_{Ax \le b} c^T x = \min_{\substack{A^T y = c \\ y \ge 0}} b^T y$$

Lemma: weak duality

$$\max_{Ax \le b} c^T x \le \min_{\substack{A^T y = c \\ y \ge 0}} b^T y$$

Proof

$s := b - Ax \ge 0$

$$0 \le s^T y = \sum_{i=1}^{n} s_i \cdot y_i$$

$$0 \le s^T y = \sum_{i=1} \underbrace{s_i \cdot y_i}_{\ge 0}$$

$$= (b - Ax)^T y = b^T y - x^T A^T y = b^T y - \underbrace{x^T c}_{c^T x} \Leftrightarrow \sum x_i c_i$$

$$\Rightarrow \quad c^T x \le b^T y$$

(Corollary: If $s^T y = 0$ then x and y are both optimal)