Linear Programs & Duality
Problem Set 7 – CS6515 (Spring 2025)

- This problem set is due on **Thursday March 13th**.
- Submission is via Gradescope.
- Your solution must be a typed pdf (e.g., via LaTeX, Markdown, etc. Anything that allows you to type math notation) – no handwritten solutions.
- Please try to make your solutions as concise and readable as possible. Most problems will have solutions that are no more than a page long. Consider using bullet points and adding space to break up large paragraphs into smaller chunks.
- There are 3 problems. Each problem is graded with 20p. **There is +1p bonus per problem** for stating (i) how long it took you to solve that problem, and (ii) how long it took you to type the answer.

19 Psychological Tricks with Rock-Paper-Scissors

Consider the game of rock-paper-scissors where both players bet \$1. We can describe the game via the following "payout matrix" $\mathbf{M} \in \mathbb{R}^{3\times 3}$, which describes the payout for player 1:

		R	P	S	Choice of player 1
Choice of player 2	R	0	+1	-1	
	P	-1	0	+1	
	S	+1	-1	0	

$$\Longrightarrow \quad \mathbf{M} = \begin{bmatrix} 0 & +1 & -1 \\ -1 & 0 & +1 \\ +1 & -1 & 0 \end{bmatrix}$$

In class, we established that the random strategy $p = (1/3, 1/3, 1/3)$ is optimal for both players.

Now player 2 decides to perform a psychological trick! Player 2 offers to pay double whenever they lose to rock. By offering to pay more, player 2 attempts to trick player 1 into picking rock more frequently. Then player 2 can play paper more frequently to beat player 1. But wait, doesn't that mean player 1 should now play scissors more frequently instead of rock? What if player 2 predicts that and picks rock? So should player 1 instead pick paper?

Attempting to predict the prediction of your opponent quickly ends up with circular arguments. Let's see if we can figure out the best strategy for player 1 via a linear program instead.

The new payout matrix is as follows (note the +2 in the first column, representing the double reward when player 1 wins with rock.)

$$\mathbf{M} = \begin{bmatrix} 0 & +1 & -1 \\ -1 & 0 & +1 \\ +2 & -1 & 0 \end{bmatrix} \tag{1}$$

Problems

1. Consider the case where player 2 agrees to pay double, whenever they lose to rock. The new payout matrix is given in (1).
   Write a linear program that models the problem of finding the optimal strategy p of players 1.
2. Solve the linear program and state the optimal strategy (i.e., list p_1, p_2, p_3 and expected payout). Which of rock/paper/scissors should player 1 pick more frequently?
3. Write the linear program that models the optimal strategy for player 2.
4. Solve the linear program and state the optimal strategy for player 2 (i.e., list p_1, p_2, p_3, and expected payout).

You can use any LP solver for subproblem 2 and 4. Use a library for your favorite programming language or use an online LP solver.

20 L1-Regression

We are given a matrix $\mathbf{A} \in \mathbb{R}^{n \times d}$ with $n \geq d$. The linear system $\mathbf{A}x = b$ is over-constrained and cannot be solved, but we can try to minimize the distance $\|\mathbf{A}x - b\|$.

Problem: Give a linear program that solves $\min_{x \in \mathbb{R}^d} \|\mathbf{A}x - b\|_1$.
Here $\|v\|_1 := \sum_i |v_i|$ is the $L1$-norm. Please add an explanation for your constraints and cost function so the TA have an easier time understanding your idea and can award partial points if there are issues.

Remark: This is also referred to as linear "$L1$-regression" or "least absolute deviation". These problems show up frequently in data analysis and machine learning. For example, each row of $\mathbf{A}$ may represent the location of a measurement, and the corresponding entry b_i is the value of that measurement. We want to find a hyperplane represented by normal-vector x that fits the data as well as possible.

Hint: $|y| \leq z$ if and only if $-z \leq y \leq z$, so you can get a bound on the absolute value by using two inequalities. It's fine if your LP introduces additional variables besides $x_1, ..., x_d$.

21 Bipartite Matching & Vertex Cover

Minimum Vertex Cover: Given an undirected graph $G = (V, E)$, find the smallest set $W \subset V$ such that every edge of G has at least one endpoint in W.

In general, the minimum vertex cover cannot be solved in polynomial time (it's NP-hard). We here want to argue that on the special case of bipartite graphs, the problem is actually solvable in poly-time. We will see that on bipartite graphs, minimum vertex cover is the dual of maximum matching. Since we have a poly-time algorithm to compute the maximum matching for bipartite matching, we can also solve vertex cover. (The size of optimal primal solution and optimal dual solution is identical by Strong Duality of linear programming, so size of max matching = size of min vertex cover).

Bipartite Matching as LP The following linear program describes bipartite matching:

$$\max_{x \in \mathbb{R}^E} \mathbf{1}^\top x$$
$$\sum_{\{u,v\} \in E} x_{uv} \leq 1 \text{ for all } v \in V$$
$$x \geq 0$$

Here the sum goes over all edges $\{u, v\} \in E$ that are incident to vertex v. Basically, we have a vector $x \in \mathbb{R}^E$ that indicates which edges we want to pick into the matching and the sum counts how many edges incident to v have been picked. In a valid matching, every vertex can be picked at most once.

In the cost function, the term $\mathbf{1}$ is a vector with all-1-entries, so $\mathbf{1}^\top x = \sum_{e \in E} x_e$, i.e., the number of edges we pick.

We won't prove it here, but on bipartite graphs, this linear program has optimal solutions where each x_e is 0 or 1, i.e., we won't have weird cases where we try to pick only half an edge $x_e = 1/2$.

Problem:

1. Write the dual linear program for the bipartite matching program.
2. Given a vertex cover $w \subset V$ (i) construct a vector y that satisfies all the constraints of the dual linear program. (ii) Verify that y indeed satisfies the constraints. (iii) Why is the cost of vector y equal to the size of the vertex cover?

Hints: For 1, it may help to write the constraints of the bipartite matching linear program as a large $|V| \times |E|$ matrix:

$$\max \mathbf{1}^\top x$$
$$\mathbf{M}x \leq \mathbf{1}$$
$$x \geq 0$$

where for edge e and vertex v we have $\mathbf{M}_{v,e} = ...$

For 2, it may help to think about what each variable and constraint of the dual represent. What are the conditions for a valid vertex cover?