Tue

$$\begin{aligned} \max\ & c^T y \\ & My \le b \end{aligned} \quad \overset{\text{Dual}}{\longleftrightarrow} \quad \begin{aligned} \min\ & b^T x \\ & M^T x = c \\ & x \ge 0 \end{aligned}$$

What if LP has a different shape?

$$\begin{aligned} \min\ & 2x_1 - 4x_2 \\ & 2x_1 + 3x_2 \le 4 \\ & 4x_1 + 5x_2 = 5 \\ & 3x_1 + 6x_2 \ge 1 \\ & x_2 \ge 0 \end{aligned} \quad \rightsquigarrow \quad \begin{aligned} \min\ & 2x_1^+ - 2x_1^- - 4x_2 \\ & 2x_1^+ - 2x_1^- + 3x_2 \le 4 \\ & 4x_1^+ - 4x_1^- + 5x_2 = 5 \\ & 3x_1^+ - 3x_1^- + 6x_2 \ge 1 \\ & x_1^+ \ge 0 \quad x_1^- \ge 0 \quad x_2 \ge 0 \end{aligned}$$

$$\rightsquigarrow \quad \begin{aligned} \min\ & 2x_1^+ - 2x_1^- - 4x_2 \\ & 2x_1^+ - 2x_1^- + 3x_2 + s_1 = 4 \\ & 4x_1^+ - 4x_1^- + 5x_2 = 5 \\ & 3x_1^+ - 3x_1^- + 6x_2 - s_2 = 1 \\ & x_1^+ \ge 0 \quad x_1^- \ge 0 \quad x_2 \ge 0 \quad s_1 \ge 0 \quad s_2 \ge 0 \end{aligned}$$

$$x_1^+ = \begin{cases} x_1 & \text{if } x_1 \ge 0 \\ 0 & \text{else} \end{cases} \qquad x_1^- = \begin{cases} -x_1 & \text{if } x_1 < 0 \\ 0 & \text{else} \end{cases}$$

$$x_1 = x_1^+ - x_1^-$$

$$\min\ (2\ \ -2\ \ -4\ \ 0\ \ 0)\, x$$

$$\begin{pmatrix} 2 & -2 & 3 & 1 & 0 \\ 4 & -4 & 5 & 0 & 0 \\ 3 & -3 & 6 & 0 & -1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 5 \\ 1 \end{pmatrix}$$

$$x \ge 0$$

Dual $\Downarrow$

$$\max\ (4\ \ 5\ \ 1)\, y$$

$$\begin{pmatrix} 2 & 4 & 3 \\ -2 & -4 & -3 \\ 3 & 5 & 6 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} y \le \begin{pmatrix} 2 \\ -2 \\ -4 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{aligned} \max\ & 4y_1 + 5y_2 + y_3 \\ & 2y_1 + 4y_2 + 3y_3 = 2 \\ & 3y_1 + 5y_2 + 6y_3 \le -4 \\ & y_1 \le 0 \quad y_3 \ge 0 \end{aligned} \quad \leftsquigarrow \quad \begin{aligned} & 2y_1 + 4y_2 + 3y_3 \le 2 \\ & -2y_1 - 4y_2 - 3y_3 \le -2 \\ & 3y_1 + 5y_2 + 6y_3 \le -4 \\ & y_1 \le 0 \quad y_3 \ge 0 \end{aligned}$$

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General case

$$\begin{aligned} \min\ & c^T x \\ & Mx \ \overset{\le}{\underset{=}{\ge}}\ b \end{aligned} \quad \leftrightsquigarrow \quad \begin{aligned} \max\ & b^T y \\ & M^T y \ \overset{\le}{\underset{=}{\ge}}\ c \end{aligned}$$

| min | max |
|---|---|
| $(Mx)_i \le b_i$ | $y_i \le 0$ |
| $(Mx)_i \ge b_i$ | $y_i \ge 0$ |

$$Mx \ \begin{smallmatrix}\le \\ \ge \\ =\end{smallmatrix} \ b \qquad M^T y \ \begin{smallmatrix}\le \\ \ge \\ =\end{smallmatrix} \ c$$

| | |
|---|---|
| $x_i \ge 0$ | $y_i \le 0$ |
| $x_j \le 0$ | $y_i \ge 0$ |
| $x_k \in \mathbb{R}$ | $y_i \in \mathbb{R}$ |

| | |
|---|---|
| $(Mx)_i \ge b_i$ | $y_i \ge 0$ |
| $(Mx)_i = b_i$ | $y_i \in \mathbb{R}$ |
| $x_i \ge 0$ | $(M^T y)_i \le c_i$ |
| $x_i \le 0$ | $(M^T y)_i \ge c_i$ |
| $x_i \in \mathbb{R}$ | $(M^T y)_i = c_i$ |

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2:34 Zero Sum Games

2 players, player A wins the amount that player B loses
and same the other way around

Eg Rock Paper Scissors

| | R | P | S |
|---|---|---|---|
| R | 0 | 1 | -1 |
| P | -1 | 0 | 1 |
| S | 1 | -1 | 0 |

Player A

Payout matrix for player A

$$M = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \quad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Player B

$$v_i = \begin{cases} 1 & \text{if player A picks } i\text{-th choice} \\ 0 & \text{else} \end{cases}$$

$$w_j = \begin{cases} 1 & \text{if player B picks } j\text{-th choice} \\ 0 & \text{else} \end{cases}$$

$$w^T M v = M_{ji}$$

where $i$ is choice PA
$j$ is choice PB

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Instead of a fixed choice, we could pick randomly.

$p = (1/2, 1/2, 0)$ $\quad P[\text{rock}] = 1/2 \quad P[\text{scissors}] = 0$
$P[\text{paper}] = 1/2$

$p$ strategy of PA
$q$ strategy of PB

Payout for player A

$$q^T M p = \sum_{ij} q_i \cdot p_j \cdot M_{ij} = \sum_{ij} P\left[\begin{smallmatrix} B \text{ picks } i \\ A \text{ picks } j \end{smallmatrix}\right] \cdot \underbrace{M_{ij}}_{\text{Payout for player A}}$$

$$= \mathbb{E}[\text{Payout for player A}$$

$q$ strategy of B

$= \mathbb{E}[$ Payout for player A if A picks according to $p$ and B picks according to $q ]$

Question: What is the best strategy for player A?

"Best" = Strategy such that no matter what strategy the opponent picks, the expected payout should be at least $T$. Maximize $T$.

"Infinite" Program

$$\max_{T, p_1, p_2, p_3} T$$
$$\sum_{i=1}^{3} p_i = 1$$
$$p_i \geq 0 \quad \text{for } i = 1,2,3$$
$$q^T M p \geq T \quad \text{for all } q \in \mathbb{R}^3, \; q_i \geq 0, \; \sum_{i=1}^{3} q_i = 1$$

Claim: $q^T M p \geq T \;\; \forall$ distributions $q \iff (Mp)_i \geq T \;\; \forall i$

Proof:

"$\Rightarrow$" $q = e_i$ standard unit vector

$$q^T M p \geq T \implies (Mp)_i \geq T$$
$$q^T M p = e_i^T (Mp) = (Mp)_i$$

"$\Leftarrow$"
$$q^T M p = \sum_i q_i \cdot (Mp)_i \geq \Big(\underbrace{\sum_i q_i}_{=1}\Big) \cdot T = T$$

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$$\max_{T, p_1, p_2, p_3} T$$
$$p_i \geq 0 \quad i = 1,2,3$$
$$\sum_i p_i = 1$$
$$(Mp)_i \geq T \quad i = 1,2,3$$

$p = (1/3, 1/3, 1/3) \quad T = 0$

Lemma: For any $x, y$ $\Rightarrow$ $c^T x \geq b^T y$

$$Mx = b$$
$$x \geq 0$$
$$M^T y \leq c$$

Corr. For any $x$ $\quad Mx = b$, $x \geq 0$ $\quad$ if $c^T x = b^T y$ then $x, y$ are optimal solutions

any $y$ $\quad M^T y \leq c$ $\quad$ for $\min c^T x$, $Mx = b$, $x \geq 0$

$\max b^T y$, $M^T y \leq c$

Dual of RPS

$$\min T'$$
$$(M^T y)_i \leq T' \quad i = 1,2,3$$
$$\sum_i y_i = 1$$
$$y_i \geq 0 \quad i = 1,2,3$$

$T' = 0$

$y = (1/3, 1/3, 1/3)$