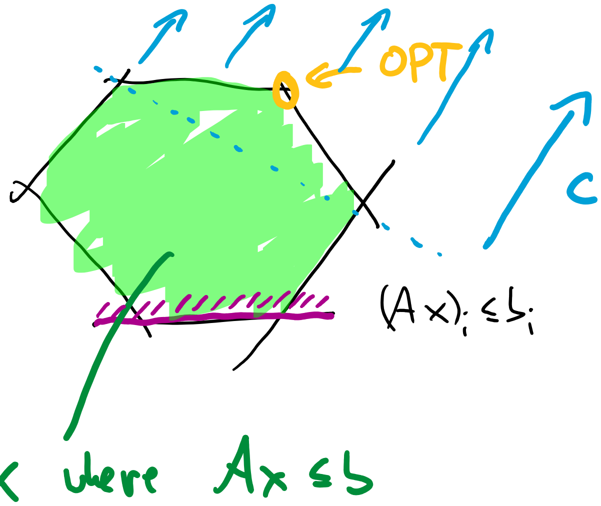
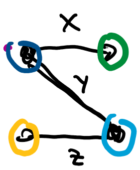
17 Simplex Algorithm

Tuesday, March 11, 2025 13:48

$$\max_{x \in \mathbb{R}^d} c^T x$$
$$Ax \le b$$
$$(Ax)_i \le b_i \quad \forall i$$

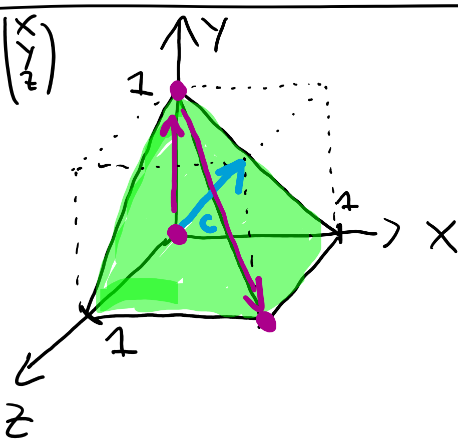


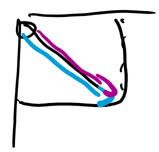


$$\max \; x+y+z = (1\;1\;1)\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

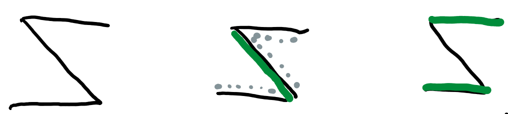
$x \ge 0$
$y \ge 0$
$z \ge 0$

$x + y \le 1$
$z + y \le 1$
$x \le 1$
$z \le 1$





$$\max \; \mathbb{1} w$$
$$w \in \mathbb{R}^E$$
$$w \ge 0$$
$$\sum_{\{u,v\} \in E} w_{uv} \le 1 \quad \forall v \in V$$



$(xyz) = (000)$ $\qquad$ (010) $\qquad$ (101)

$+(1\;-1\;1)$

$c^T(xyz) = 0$ $\qquad$ $= 1$ $\qquad$ $= 2$

$c^T(1\;-1\;1)$
$= 1$

Idea: Start at some vertex of polytope
Go along $\underbrace{\text{edges}}_{\text{of the polytope}}$ from one vertex to another, always increasing the cost

Given $2n$ constraints
how many vertices can there be?
$x \in \mathbb{R}^n \quad 0 \le x_i \le 1 \quad i = 1 \ldots n$

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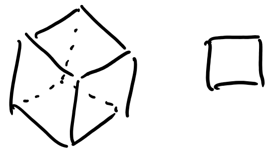
hypercube vertices are $\{0,1\}^n$

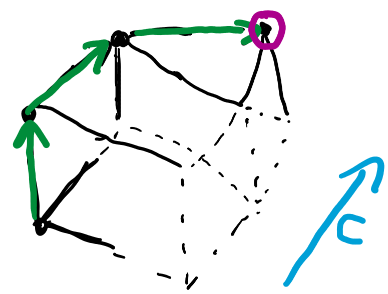


Def: Given a polytope via $Ax \le b$
the polytope graph is $G = (V, E)$
V = vertices of the polytope
E = edges of the polytope

Task: Find path from start vertex to optimal vertex.

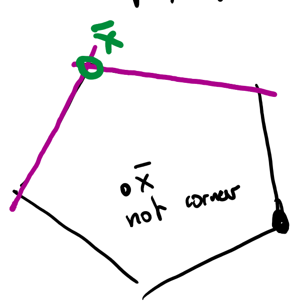
Cannot simply create graph because it can be exponential size, see hypercube.

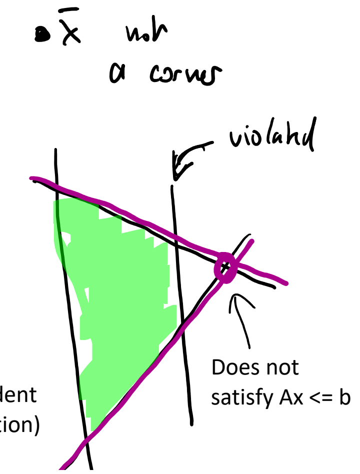
Input: A, b, c ($\max c^T x \quad Ax \le b$)
- Start at vertex
- Compute edges for this vertex
- Pick one edge that increases the cost
- go along that edge until we hit another vertex
- terminate if no edge improves the cost

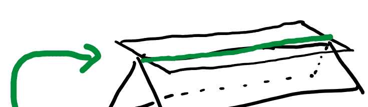


Def: Given $Ax \le b$, $A \in \mathbb{R}^{n \times d}$
then some $\bar{x} \in \mathbb{R}^n$ is a corner/vertex of the polytope
if and only if
- $A\bar{x} \le b$
- $(A\bar{x})_i = b_i$ for d many linear independent constraints

$A x \le b$







not a corner because the constraints are linearly dependent
(Linear system A_B x = b_B does not have a unique solution)

Observation:

A set $B \subseteq \{1 \dots n\}$

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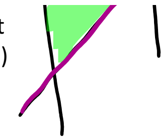
(Linear system A_B x = b_B does not have a unique solution)



satisfy Ax <= b

A set $B \subseteq \{1 \ldots n\}$

is an alternative description of a corner

if $A_B :=$ subset of rows of A, with index in B

$$\bar{x} = A_B^{-1} b_B, \qquad A_B \bar{x} = b_B$$

and $\bar{x}$ is a corner (ie $A\bar{x} \le b$)

Given A, b $\quad \{x \in \mathbb{R}^d \mid Ax \le b\} \leftarrow$ polytope

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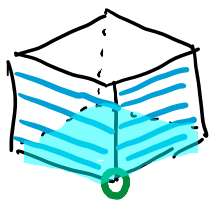
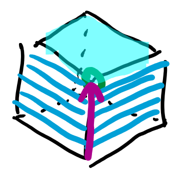
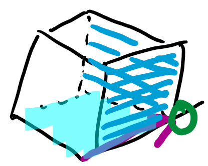
What are edges?



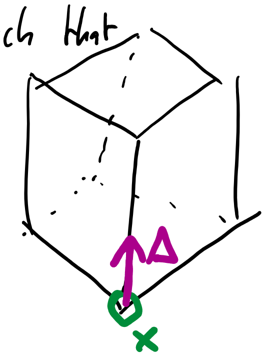




Going from vertex to vertex means to replace one index $i \in B$ by another index j

Each $i \in B$ we can remove corresponds to an edge

How to decide which $i \in B$/edge to pick such that moving along edge improves $c^T x$?



$$x^{new} \leftarrow x + \lambda \cdot \Delta$$

$\lambda \in \mathbb{R}$, $\lambda \ge 0$ $\qquad$ $\Delta \in \mathbb{R}^d$ direction of edge

We need that $c^T x^{new} > c^T x$

$$c^T x^{new} = c^T(x + \lambda \cdot \Delta) = c^T x + \lambda \cdot \underbrace{c^T \Delta}_{>0}$$

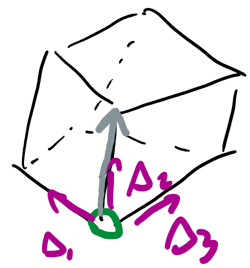
$\Leftrightarrow \quad c^T \Delta > 0$

Given B, $i \in B$ what is the direction Δ of the edge represented by B, $i \in B$?

Algorithm so far

Input: A, b, x $\quad A \in \mathbb{R}^{n \times d}$, $b \in \mathbb{R}^n$, $x \in \mathbb{R}^d$, x is vertex

$B \subseteq \{1 \dots n\}$ such that A_B is full rank and $A_B x = b_B$



compute Δ_i for $i \in B$ direction of edges (TODO)

pick i such that $c^T \Delta_i > 0$

Pick largest λ such that $A(x + \lambda \cdot \Delta_i) \leq b$

$x \leftarrow x + \lambda \cdot \Delta_i$

$B \leftarrow (B \setminus \{i\}) \cup \{j\}$ where j is such that $(A(x + \lambda \Delta_i))_j = b_j$

but $(A x)_j < b_j$