Proof Thm 1

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Proof of Thm 1)

$$\|u-v\|^2 = \|u\|^2 - 2\langle u, v\rangle + \|v\|^2$$

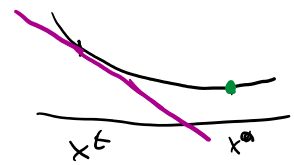
First, let us analyze $\|x^t - x^*\|^2$

$$\|x^{t+1} - x^*\|^2 = \|x^t - x^* - \eta \nabla f(x^t)\|^2 = \|x^t - x^*\|^2 - 2\eta \underbrace{\nabla f(x^t)^T (x^t - x^*)}_{\geq f(x^t) - f(x^*)} + \eta^2 \underbrace{\|\nabla f(x^t)\|^2}_{\leq L^2}$$

$$\Updownarrow$$

$$f(x^*) \geq f(x^t) + \nabla f(x^t)^T (x^* - x^t)$$

$$\leq \|x^t - x^*\|^2 - 2\eta\left(f(x^t) - f(x^*)\right) + \eta^2 L^2$$



$$\Rightarrow \quad f(x^t) - f(x^*) \leq \frac{\|x^t - x^*\|^2 - \|x^{t+1} - x^*\|^2 + \eta^2 L^2}{2\eta}$$

$$\Rightarrow \quad \left(\frac{\sum_{t=0}^{T-1} f(x^t)}{T}\right) - f(x^*) \leq \frac{\sum_{t=0}^{T-1} \frac{\|x^t - x^*\|^2 - \|x^{t+1} - x^*\|^2 + \eta^2 L^2}{2\eta}}{T}$$

$$\leq \frac{\|x^0 - x^*\|^2 \cancel{- \|x^T - x^*\|^2}}{2\eta T} + \frac{\eta L^2}{2}$$