23 Schwartz-Zippel Lemma & Matching

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Given 2 functions $f(x)$ $g(x)$

Do they compute the same thing?

$f(x) = g(x) \quad \forall x$

$h(x) = f(x) - g(x)$ Is $h(x) = 0 \quad \forall x$?

Schwartz-Zippel Lemma:

Let h be some non-zero polynomial ($\exists x$ such that $h(x) \neq 0$)

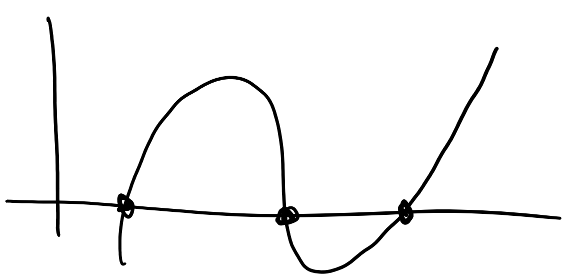
$h : \mathbb{F}^n \to \mathbb{F}$ (n inputs from some field $\mathbb{F}$)

For $x_1, x_2 \dots x_n$ uniformly at random

then $P(h(x_1, x_2, \dots, x_n) = 0) \leq \frac{d}{|\mathbb{F}|}$ ← degree of h

← # elements in the field

$n=1$



$h : \mathbb{F} \to \mathbb{F}$ of degree d can have at most d inputs where $h(x) = 0$

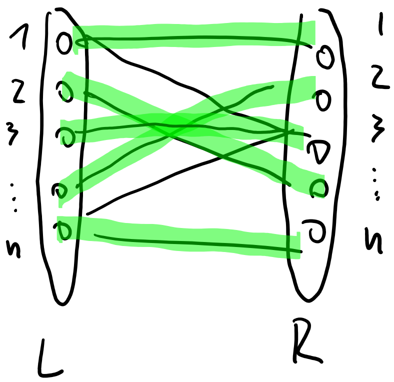
$n=1$ $x^3 - 3x^2 + 5x - 1$

$d=3$

For $n>1$ do induction over n.

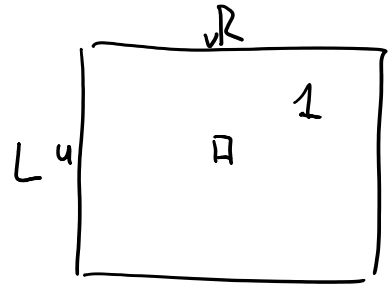
$n=2$ $x^{3}y^{2} - 3xy + 10$

$d = \boxed{5}$



$$A_{u,v} = \begin{cases} 1 & \text{if } \{u,v\} \\ 0 & \text{else} \end{cases}$$

$u \in L$ $\quad \in R$



$$\det(A) = \sum_{\sigma \in S_n} \Big(\underbrace{\operatorname{sign}(\sigma)}_{\in \{+1,-1\}} \cdot \prod_{i=1}^{n} A_{i,\sigma(i)} \Big)$$

S_n: set of all permutations on $\{1 \dots n\}$

$$\prod_{i=1}^{n} A_{i,\sigma(i)} = \begin{cases} 1 & \text{if } \{i,\sigma(i)\} \in E \quad \forall i = 1 \dots n \\ 0 & \text{else} \end{cases} = \begin{cases} 1 & \sigma \text{ is valid matching} \\ 0 & \end{cases}$$

$\sigma : \{1 \dots n\} \longrightarrow \{1 \dots n\}$

1 to 1 connection from $i \longleftrightarrow \sigma(i)$

σ is like a "hypothetical" matching (it matches vertex i on the left with $\sigma(i)$ on the right) an the product $\prod A_{i,\sigma(i)}$ tests if all the needed edges exist to ~~form~~ this edges exist to ~~form~~ this matching.

Observation:

If there is no matching of size n

then $\prod_{i=1}^{n} A_{i,\sigma(i)} = 0 \quad \forall \sigma \in S_n$

$\Rightarrow \sum \text{sign}(\sigma) \prod A_{i,\sigma(i)} = 0$

$\Rightarrow \det(A) = 0 \longrightarrow$

Contrapositive:

If $\det(A) \neq 0$

then there is a matching of size n

("perfect matching" because every vertex is matched)

Idea to prevent cancellations:

Plug in random numbers $A_{uv} = \begin{cases} r_{uv} & \text{if } \{u,v\} \in E \\ 0 & \text{else} \end{cases}$

each r_e is uniformly independently sampled random number.

Claim: Given $G = (L, R, E)$ and the randomized adjacency matrix A, we have with high probability that

$\det(A) \neq 0 \iff G$ has a perfect matching

Proof: If $\det(A) \neq 0$

$$\det(A) = \sum_{\sigma} \left(\text{sign}(\sigma) \prod_{i=1}^{n} A_{i,\sigma(i)} \right)$$

$\Rightarrow \exists \sigma$ where $\prod_{i=1}^{n} A_{i,\sigma(i)} \neq 0$

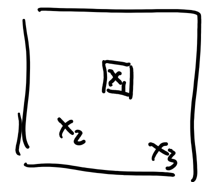
$\Rightarrow \exists \sigma$ where $A_{i,\sigma(i)} \neq 0 \ \forall i = 1 \ldots n$

$\Rightarrow$ [illegible] of size n

$\Rightarrow \exists \sigma$ where $A_{i,\sigma(i)} \neq 0 \ \forall i = 1 \dots n$

$\Rightarrow M = \{ \{i, \sigma(i)\} \mid i = 1 \dots n \}$ is a valid matching of size n

If there is a perfect matching $M \subseteq E$



$$h(x_1 x_2 x_3 \dots x_{|E|}) = \det(A_x)$$

adjacency matrix with x_i as entry for each edge

h is a non-zero polynomial because for input $x_e = \begin{cases} 1 & \text{if } e \in M \\ 0 & \text{else} \end{cases}$

we have $h(x_1 \dots x_{|E|}) = \det(A_x) = \sum_\sigma \operatorname{sign}(\sigma) \underbrace{\prod_{i=1}^{n} (A_x)_{i,\sigma(i)}}_{= \begin{cases} 1 & \text{if } \sigma \text{ represents } M \\ 0 & \text{else} \end{cases}}$

$= \operatorname{sign}(\sigma) \cdot 1$ for σ representing M

$\neq 0$

h is of degree n because the product have n terms

$\Rightarrow$ by Schwartz Zippel lemma we have for random $x_1 \dots x_{|E|}$ that $P(h(x_1 \dots x_{|E|}) = 0) \leq \frac{n}{|\mathbb{F}|} << \varepsilon$ by picking large enough field

$\Rightarrow$ If $\exists$ perfect matching then $P(h(\dots) \neq 0) \geq 1 - \varepsilon$

Algo: - construct adjacency matrix for random numbers for each edge
- compute determinant
- if 0 whp no perfect matching
- if det $\neq 0$ there is perfect matching