24 One Sided Error, Amplification, k-path

Tuesday, April 15, 2025 13:53

Deck of cards

Is there any red card?

Algo:

```
Pick a uniformly random card
if card is red
    return true
else
    return false
```

If algo. returns true
then the answer is guaranteed to be correct

if returns false
then answer is incorrect with some "failure probability"

$$\frac{n-r}{n}$$

Def: A randomized algorithm has "one-sided error" if the boolean output is guaranteed to be correct for one type of answer (eg "true" is always correct but "false" could be wrong with some failure probability < 1)

If we have some algo. with one-sided error (let's assume "true" is correct) with failure probability p.

```
for i=1...T
    if algo returns true
        return true
return false
```

P(new algo incorrectly returns false)

$$= \prod_{i=1}^{T} P(\text{ith call returned false incorrectly})$$

$$= p^T$$

... $T=20$ then new algo is only 20 times slower return value

Example: If $p = 1/2$ and $T = 20$ then new also is only 20 times slower

but failure probability is $\frac{1}{2^{20}} \approx \frac{1}{1 \text{ million}}$

This is referred to as "amplification"

K- path

Input: $G = (V, E)$ directed graph

Output: Yes/No if there exists a path of length k

without cycles

For any $k = n^{\varepsilon}$, constant ε, this is NP-hard $n^{0.00001}$

What if k is constant? Can we have a fast/poly-time algorithm?

Simple/Slow try all $\binom{n}{k} \leq n^k$ $v_1 v_2 \ldots v_k$ vertex k-tuples

and check if (v_i, v_{i+1}) is a valid edge

$\Rightarrow O(n^k)$

Nicer would be a time complexity like $O(|E|)$ $O(m)$ time also

eg $O(2^k \cdot m) = O(m)$ if k is constant

"FPT" fixed parameter tractable

Technique "Color Coding"

- Assign each vertex a random color, a number from $\{1, 2, 3, \ldots k\}$

- Remove all edges from G but keep those that connect

color i to color $i+1$

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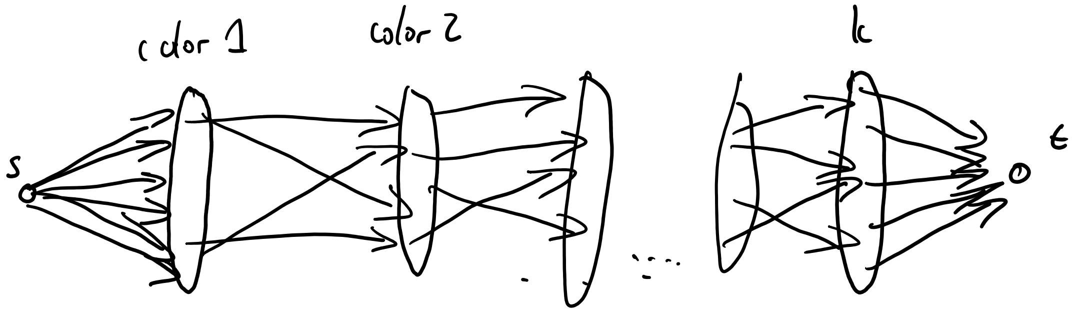
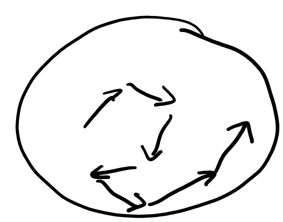


- add vertex s, t, connect s to all color 1 vertices, connect all color k vertices to t

If s can reach t in this new graph, then the original input graph has a k-path.

One-sided error, "true" is correct
"false" might be incorrect.



failure probability
- we fail if there is a k-path but it did not receive color $1, 2, 3 \ldots k$ in the same order as the path visits the vertices.

$$P(\text{failure}) = 1 - P(\text{any of the } k\text{-paths survives})$$

$$\leq 1 - P(\text{the } \overbrace{\quad}^{\text{first } k\text{-path}} \text{ survived}) = 1 - \prod_{i=1}^{k} P(i\text{-th vertex of } k\text{-path gets color } i)$$

$$= 1 - \frac{1}{k^k}$$

Let's repeat this algorithm $\underbrace{\text{some}}_{\text{with new coloring}}$ $c \cdot k^k$ times c is a constant.

$c \cdot k^k$

"with new coloring"

$$P(\text{all } c \cdot k^k \text{ repetitions fail}) = \prod_{i=1}^{c \cdot k^k} P(\text{the i-th repetion fails})$$

$$\leq \left(1 - \frac{1}{k^k}\right)^{c \cdot k^k} = \left(\left(1 - \frac{1}{k^k}\right)^{k^k}\right)^c$$

$$\left(1 - \frac{1}{n}\right)^n \leq \frac{1}{e}$$

$$\lim_{n \to \infty} \left(1 - \frac{1}{n}\right)^n = \frac{1}{e}$$

$$\leq \frac{1}{e^c}$$

eg if we pick $c = \ln(10^6)$ then failure $\leq \frac{1}{1 \text{ million}}$

Time complexity $O(k^k \cdot |E|) = O(|E|)$

k^k: exponential in k

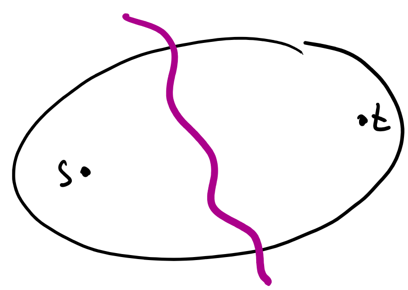
$=$: if k is constant

Remark $O(2^k \cdot |E|)$ exists, similar idea.

(NP-hard for $k = n^{0.01}$)

(Global) Min-Cut

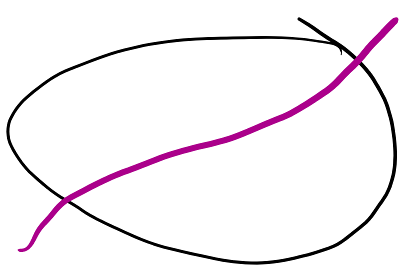
Undirected graph



st-min-cut

GuessMinCut($G = (V, E)$)

- pick uniformly random edge $e \in E$
- contract edge e
  keep multi-edges
- repeat until 2 vertices left
- Let L = vertices contracted into first remaining vertex
  R = vertices contracted into other remaining vertex



- any cut that splits graph into 2 pieces
- find smallest such cut (smallest set of edges)

R = vertices contracted into [illegible] remaining [illegible]