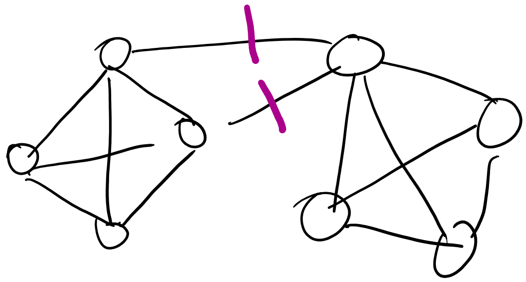
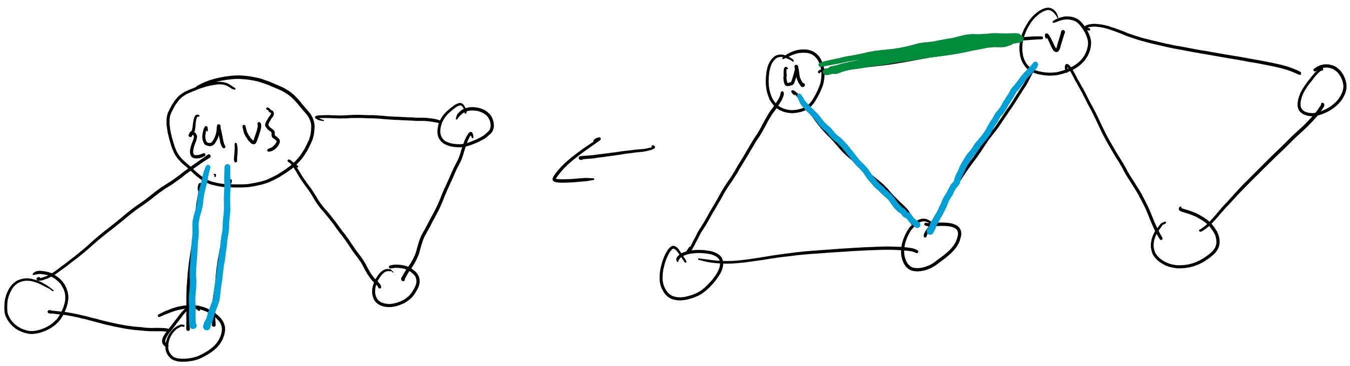
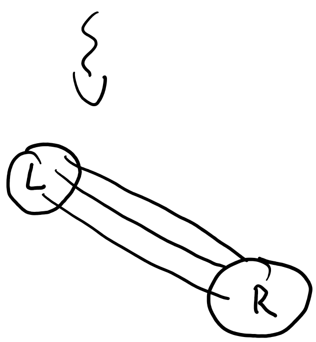
25 (global) min cut

Thursday, April 17, 2025 13:59



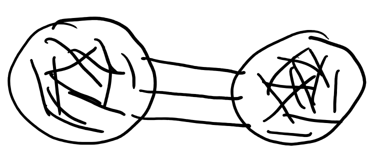
Find smallest edge set
such that their removal
splits the graph into 2 components.





$L, R \subseteq V$

Repeated merging/contraction produces a cut, but is that the minimum cut?



Observation: The min cut survives $\Leftrightarrow$ we never contract
an edge of the min cut

Consider some min cut $C \subseteq E$

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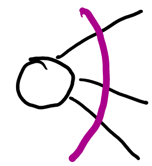
Algorithm returns C if it never picks $e \in C$

Claim: For uniform $e \in E$ $\quad P(e \in C) \leq \frac{2}{n}$ $\qquad n = |V|$

Proof: Let k is the size of min cut



$\Rightarrow$ every vertex has degree $\geq k$

$$\Rightarrow |E| = \frac{\sum_{v \in V} \deg(v)}{2} \geq \frac{nk}{2}$$

$$\Rightarrow P(e \in C) = \frac{|C|}{|E|} = \frac{k}{|E|} \leq \frac{k}{\frac{nk}{2}} = \frac{2}{n}$$

Claim: $P($the algorithm never picks $e \in C) = P($algorithm returns min cut$)$

$$\geq \frac{2}{n(n-1)}$$

Proof: Let $e_1 e_2 \ldots$ be the edges picked by the algorithm

$$P(e_1 \notin C, e_2 \notin C, \ldots) = \prod_{i=1}^{n-2} P(e_i \notin C \mid e_1 e_2 \ldots e_{i-1} \notin C)$$

$$= \prod_{i=1}^{n-2} 1 - P(e_i \in C \mid e_1, e_2, \ldots e_{i-1} \notin C)$$

$$\geq \prod_{i=1}^{n-2} 1 - \frac{2}{n-i+1}$$

$$= \prod_{i=1}^{n-2} \frac{n-i-1}{n-i+1} = \frac{\prod_{i=1}^{n-2} n-i-1}{\prod_{i=1}^{n-2} n-i+1}$$

$$= \frac{\prod_{i=3}^{n} n-i+1}{\prod_{i=1}^{n-2} n-i+1} = \frac{1}{n-1+1} \cdot \frac{1}{n-2+1} \cdot (n-(n-1)+1)$$

$$= \frac{\prod_{i=3} \cdots}{\prod_{i=1}^{n-2} n-i+1} = \frac{1}{n-1+1} \cdot \frac{1}{n-2+1} \cdot (n-(n-1)+1) \cdot (n-n+1)$$

$$= \frac{2}{n(n-1)}$$

If we repeat above algorithm $O(n^2 \log n)$ times and return the smallest of the produced cuts then with probability $\underbrace{1 - \frac{1}{n^{100}}}_{\text{"with high probability"}}$ we return the min cut.

$$P(\text{we do not return min cut}) = P(\text{none of the } O(n^2 \log n) \text{ repetitions have found the min cut})$$

$$\left(1 - \frac{1}{x}\right)^x \leq \frac{1}{e}$$

$$= \left(1 - \frac{2}{n(n-1)}\right)^{O(n^2 \log n)}$$

$$= \left(\left(1 - \frac{1}{\left(\frac{n(n-1)}{2}\right)}\right)^{\frac{n(n-1)}{2}}\right)^{O(\log n)}$$

$$\leq \left(\frac{1}{e}\right)^{O(\log n)} = \frac{1}{n^{O(1)}}$$

$$\leq \frac{1}{n^{100}} \text{ if}$$

we pick large enough constant for the number of repetitions

$O(n^2 \log n)$

Time Complexity

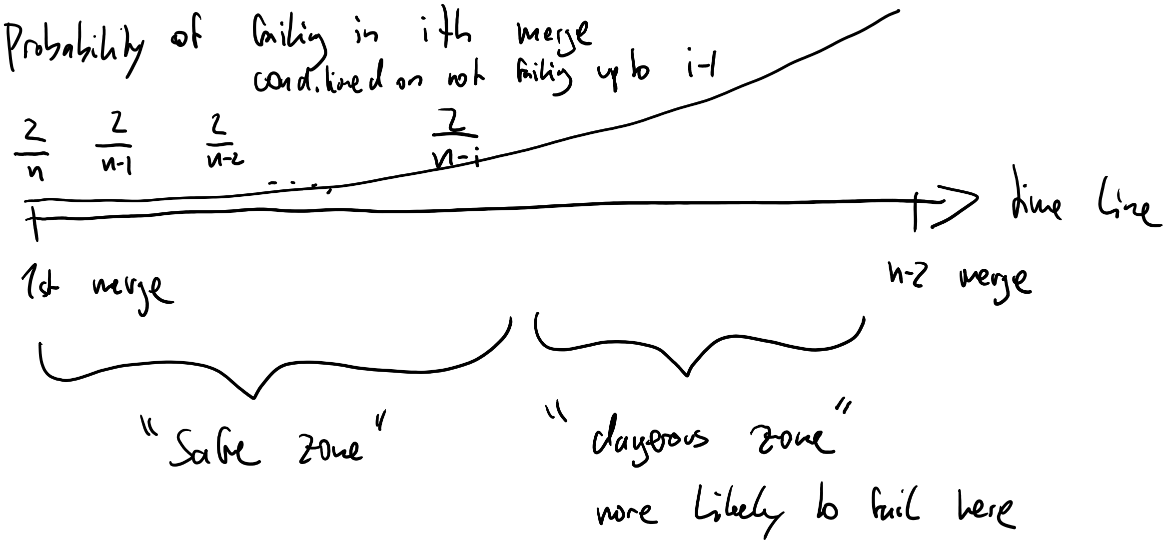
$O(n^2)$ (n-2 merges and each

Time Complexity

running Algo once is $O(n^2)$ ($n-2$ merges and each merge takes $O(n)$ time)

$\Rightarrow O(n^2 \log n)$ repetitions leads to $O(n^4 \log n)$ time.

Probability of failing in ith merge
conditioned on not failing up to i-1



Better Min Cut ($G=(V,E)$)   if $|V|=2$, then that's the cut.

$n = |V|$

while $|V| > \frac{n-1}{2}$ // previous algo kept merging until 2 vertices left

- uniform e
- merge/contract e

New algo keeps merging until # vertices has halved.

Let $\overline{G}$ is new graph after these merges

for $i = 1, \ldots 4$

x_i = BetterMinCut ($\overline{G}$)

return smallest cut among x_1, x_2, x_3, x_4

$$T(n) = 4 \cdot T\left(\frac{n}{2}\right) + O(n^2) = O(n^2 \log n)$$

Claim: $P($we pick $e \in C$ during $) \leq \frac{3}{4}$

$P($cut C survives -phase$) \geq \frac{1}{[illegible]}$

Claim: P(we pick $e \in C$ during [redacted]) ... $\frac{}{4}$

$$P(\text{cut } C \text{ survives [redacted]-phase}) \geq \frac{1}{4}$$

Proof: $$\prod_{i=1}^{\frac{n-1}{2}} 1 - \frac{2}{n-i+1} = \prod_{i=1}^{\frac{n-1}{2}} \frac{n-i-1}{n-i+1} = \frac{\prod_{i=3}^{\frac{n-1}{2}+2} n-i+1}{\prod_{i=1}^{\frac{n-1}{2}} n-i+1} \approx \frac{1}{n(n-1)} \cdot \frac{(n-1)}{2} \cdot \frac{(n+1)}{2}$$

$$\geq \frac{1}{4}$$

Claim: $P(\text{algo returns } C) = \Omega\left(\frac{1}{\log n}\right)$

Proof: $$p(n) := \underbrace{P(\text{no } e \in C \text{ is contracted in [redacted]})}_{C \text{ still exists in } \overline{G}} \cdot P\left(\begin{array}{c}\text{one of } X_1 X_2 X_3 X_4 \\ \text{is cut } C\end{array} \middle| \begin{array}{c} C \text{ is still} \\ \text{exists in } \overline{G}\end{array}\right)$$

$$\geq \frac{1}{4} \cdot \left(1 - P(\text{none of } X_1 .. X_4 \text{ are cut } C \mid C \text{ is still in } \overline{G})\right)$$

$$= \frac{1}{4} \cdot \left(1 - P(X_1 \neq C)^4\right) = \frac{1}{4} \cdot \left(1 - (1 - P(X_1 = C))^4\right)$$

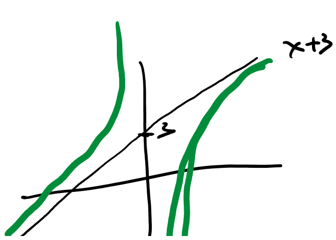
$$= \frac{1}{4}\left(1 - \left(1 - p\left(\tfrac{n}{2}\right)\right)^4\right)$$

$$p(n) \geq \frac{1}{4}\left(1 - \left(1 - p\left(\tfrac{n}{2}\right)\right)^4\right)$$

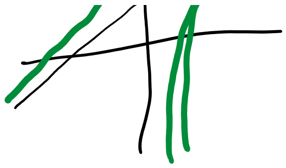
Claim:
$$p(n) \geq \frac{1}{3\log_2(n)}$$

$$d(k) = \frac{1}{p(2^k)}$$

$$d(k) = \frac{4}{1 - \left(1 - \frac{1}{d(k-1)}\right)^4} = \frac{4\, d(k-1)^4}{d(k-1)^4 - (d(k-1) - 1)^4} \leq d(k-1) + 3$$



$$\frac{4x^4}{x^4 - (x-1)^4} \leq x + 3$$



$$d(k) \le 3k$$

$$p(n) \ge \frac{1}{3\log_2(n)}$$