

Continuous Random Variable

X is a continuous random variable if it assigns a real number $X(s)$ to an uncountable event space's events.

Cumulative distribution function (cdf)

$$F_X(x) = P(X \leq x), -\infty < x < \infty$$

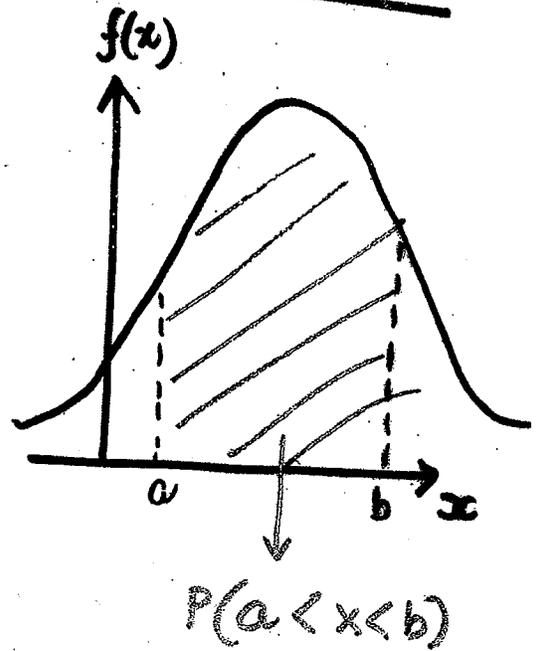
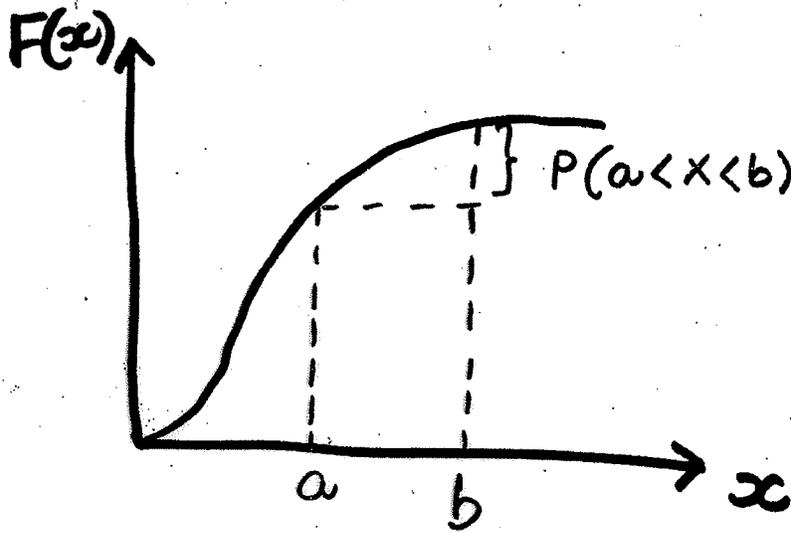
F_X is a smooth distribution function unlike in the case of discrete r.v.s.

Probability density function (pdf)

$$f(x) = \frac{dF(x)}{dx}$$

$$F(x) = \int_{-\infty}^x f(t) dt$$

Relation between pdf & cdf



$$\begin{aligned} P(a < X < b) &= \int_a^b f(t) dt \\ &= \int_{-\infty}^b f(t) dt - \int_{-\infty}^a f(t) dt \\ &= F(b) - F(a) \end{aligned}$$

Exponential Distribution

~ If job arrivals follow a Poisson process, then inter-arrival time follows Exponential distribution.

Example: • Time to failure of a component
• Service time at a server in a queuing network.

$$F(x) = 1 - e^{-\lambda x}, \quad x > 0.$$
$$f(x) = \lambda e^{-\lambda x}, \quad x > 0$$
$$X \sim \text{EXP}(\lambda)$$

Memoryless property

$$P(Y \leq y | X > t) = P(X < y)$$

independent of t .

Given a component has survived till time t , what is the probability it will ~~survive~~ ^{fail} ~~till~~ ⁱⁿ $t+y$.

$$Y = X - t$$

Exponential Distribution

Derivation of memoryless property

$$P(Y \leq y | X > t) = P(X \leq t+y | X > t)$$

$$\boxed{Y = X - t} = P(t < X \leq t+y)$$

$$= \frac{F(t+y) - F(t)}{1 - F(t)} = \frac{1 - e^{-\lambda(t+y)} - (1 - e^{-\lambda t})}{1 - (1 - e^{-\lambda t})}$$
$$= \frac{e^{-\lambda t} - e^{-\lambda(t+y)}}{e^{-\lambda t}} = P(X \leq y)$$

Suppose failure rate is λ (constant)

P (component will fail in time t)

$= 1 - P$ (No failure in time t)

$= 1 - P(N_t = 0$ in Poisson process)

$$= 1 - e^{-\lambda t} \frac{(\lambda t)^0}{0!}$$

$= 1 - e^{-\lambda t} \longrightarrow$ Exponential dist

Reliability & Failure Rate

$$R(t) = 1 - \text{Prob. (component fails in time } t)$$
$$= 1 - F(t)$$

$$R'(t) = -f(t) \quad f(t) = F'(t)$$

Say, component survived till time t .

$P(\text{component will not survive an addl. time } x \mid \text{survived till time } t)$

$$= \frac{P(t < x < t+x)}{P(x > t)}$$

$$= \frac{F(t+x) - F(t)}{R(t)}$$

$h(t)$: instantaneous failure rate

$$= \lim_{x \rightarrow 0} \frac{F(t+x) - F(t)}{R(t)} \cdot \frac{1}{x}$$

$$= \frac{1}{R(t)} \lim_{x \rightarrow 0} \frac{F(t+x) - F(t)}{x} = \frac{f(t)}{R(t)}$$
$$= \frac{1}{R(t)} F'(t)$$



Failure Rate

For exponential distribution

$$\text{failure rate } h(t) = \frac{f(t)}{R(t)}$$

$$= \frac{\lambda e^{-\lambda t}}{e^{-\lambda t}} = \lambda$$

$h(t) \Delta t$: Given that component has survived till 't', what is the probability it will fail in $(t, t + \Delta t]$.

$f(t) \Delta t$: Unconditional probability that the component will fail in $(t, t + \Delta t]$

$h(t) \not\equiv f(t)$?

$$R(t) = e^{-\int_0^t h(x) dx}$$

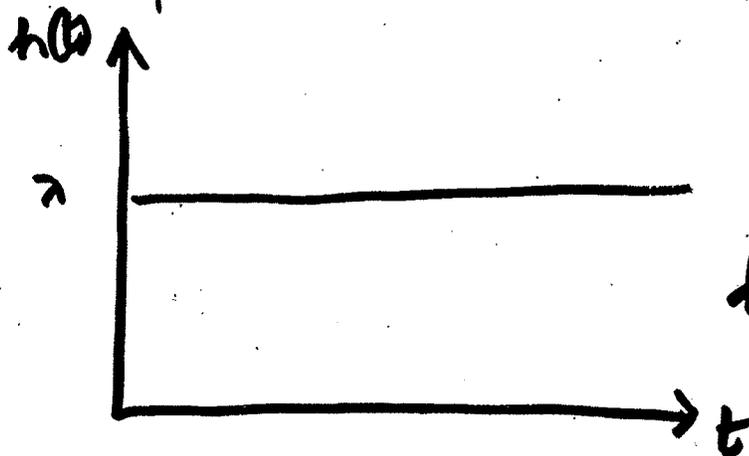
For exponential $\left[\begin{aligned} &= e^{-\int_0^t \lambda dx} \\ &= e^{-\lambda t} \end{aligned} \right.$

$$\begin{aligned} R(t) &= 1 - F(t) \\ &= 1 - [1 - e^{-\lambda t}] \\ &= e^{-\lambda t} \end{aligned}$$

⑥

Failure Rate

With exponential distribution

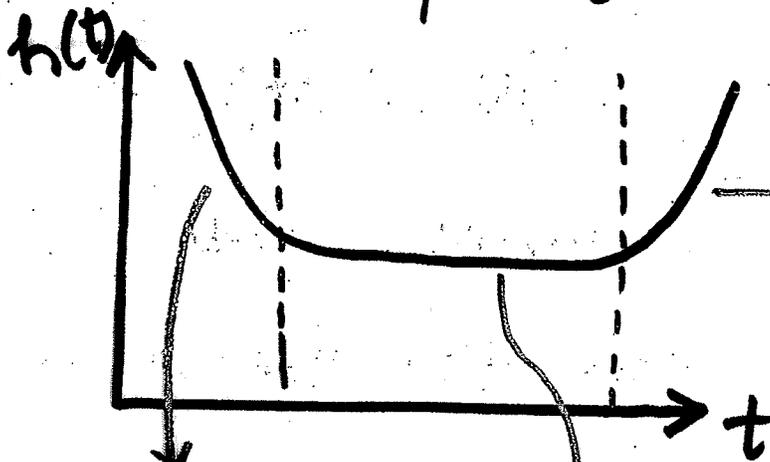


$$f(t) = \lambda e^{-\lambda t}$$

$$R(t) = e^{-\lambda t}$$

$$h(t) = \frac{f(t)}{R(t)} = \lambda$$

Typical computer failure rate distribution



Increasing failure rate
(Wear-out Phase)

Decreasing failure rate
(Burn-in period)

Constant failure rate
(Useful life)

Example problem #1.

Hypoexponential Distribution

A process goes through a sequence of phases. Time it spends at each phase is independent and exponentially distributed. Overall time spent by the process in all phases follows hypoexponential distribution

parameters = # stages

2-stage hypoexponential: $X \sim \text{HYPO}(\lambda_1, \lambda_2)$

$$f(t) = \frac{\lambda_1 \lambda_2}{\lambda_1 - \lambda_2} (e^{-\lambda_1 t} - e^{-\lambda_2 t}), t > 0$$

Denominator should be $\lambda_2 - \lambda_1$

$$F(t) = ? \int_0^t f(x) dx$$

$$h(t) = ? \quad \frac{f(t)}{R(t)} = \frac{f(t)}{1 - F(t)}$$

Which stage of a component's lifetime can this characterize?